Homework Set #3 (Due Tuesday, May 3rd)

1) Consider the traffic flow problem, which obeys the continuity equation for density of vehicles $\rho(x, t)$,

$$
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0,
$$

(1)

where $v(x, t) = v_{\text{max}}(1 - \rho/\rho_{\text{max}})$ is the velocity field, with $v_{\text{max}}$ the speed limit, and $\rho_{\text{max}}$ the density corresponding to vehicles filling space bumper to bumper.

a) Explain why such a dependence of $v$ on $\rho$ makes physical sense. Calculate the wave speed $c(\rho)$ and compare with $v(\rho)$, explain their difference.

b) Impose periodic boundary conditions (as in a race track) and a square wave initial condition,

$$
\rho(x, 0) = \rho_{\text{max}}, \quad -L/4 < x < 0,
$$

(2)

and zero otherwise, where $L$ is the size of the track, $-L/2 \leq x \leq L/2$.

c) Solve the PDE by evolving $\rho$ using the methods FTCS and Lax. Figure out reasonable values for $v_{\text{max}}$, $\rho_{\text{max}}$, $L$, and time of evolution (e.g. so that cars go around at least once). Make a choice of resolution of your spatial grid, and then fix your time step appropriately. Explain the logic behind your choices.

d) Make a 3D plot showing the density $\rho$ as a function of space and time (or 2D plots with $\rho(x)$ for different times). Compare the results of the two methods and explain the main features of the solution.