Homework Set #4 (Due Thursday, December 14th)

1) Consider the following linear PDE (with $a$, $b$ constants and $n$ integer),

$$
\frac{\partial f}{\partial t} + b x \frac{\partial f}{\partial x} = a x^n.
$$

a) Solve Eq. (1) by the method of characteristics. Express your solution in terms of the initial condition $f(x, t = 0) = f_0(x)$.

b) Plot your solution as a function of $x$ at different times $t = 0, 1, 2$, for $f_0(x) = \exp(-(x-1)^2)$, $b = n = 1$. Consider two cases, $a = 0$ and $a = 0.05$. Explain what you see in the plots.

2) Consider the KdV equation in one dimension,

$$
\frac{\partial f}{\partial t} + f \frac{\partial f}{\partial x} + \alpha \frac{\partial^3 f}{\partial x^3} = 0,
$$

where $-L \leq x \leq L$ and periodic boundary conditions are imposed, the initial condition corresponds to a small amplitude cosine wave, $f(x, t = 0) = (1/8) \cos(\pi x/L)$.

a) Solve the KdV equation using the Galerkin method, as discussed in class. In doing so, you convert the PDE into a system of coupled ODE’s which can be solved by your Runge-Kutta solver. Use plane waves as modes to expand your solution, including up to $M = 20$ modes, and evolving from $t = 0$ to $t = 200$. Use $\alpha = 1$ for your numerical solution, and the standard inner product.

b) Make plots of $f(x, t)$ as a function of $x$ for times $t = 0, 20, 40, 80, 120, 200$.

c) Calculate and make a plot of the power spectrum for the timesteps in b).

d) Explain what you see in the plots in b) and c). What would happen if we decrease $\alpha$?

3) Consider the Ising model in a square lattice in two dimensions of size $L = 50$ with free boundary conditions. Normalize the maximum magnetization to unity (i.e. all spins up).
a) Use the Metropolis algorithm to find the equilibrium state at a given temperature, choosing the number of Monte Carlo steps per spin $t_n = n/L^2$ such that a “cold” initial condition (e.g. all spins up) and a “hot” initial condition (all spins random) gives approximately the same magnetization after $n$ flip proposals.

b) Calculate the magnetization as a function of temperature and make a plot of it, comparing with the analytic prediction of the critical temperature $k_B T_c = 2.269J$. At each temperature calculate the magnetization by averaging over 5 seeds of initial “hot” conditions, each of them evolved through $t_n$ steps of the chain.