We considered the two-body problem in Newtonian gravity, now we will take a look at what happens in GR with the orbits of planets. In this case, we will not solve the two-body problem, but rather the motion of a test particle (planet) under the influence of a source (sun); equivalent to the two-body problem when one mass dominates over the other. We shall see that orbits are no longer closed in GR, and precession results. In the homework we will explore this numerically (as opposed to doing small perturbations as we'll do here).

Before we discuss GR, let me briefly summarize special relativity (SR).

In Newtonian mechanics, we have addition of velocities (i.e. if system S' moves with respect to S at speed \( u \), and a particle in S' moves with velocity \( v' \) \( \Rightarrow \) in S that particle has \( v = u + v' \)). The question then arises of whether light respects this addition property. If so, measuring the speed of light from a distant source when the earth moves in one direction, and then six months later when it moves in the opposite direction will give different answers (related to how fast the earth is moving) — the absence of this effect (e.g. see the Michelson-Morley experiment) demonstrates that the speed of light is independent of the motion of the source.

Einstein based SR in 2 postulates:

i) All physical laws are the same in any inertial frame (Same as Newtonian case).

ii) The speed of light has an absolute value \( c = 3 \times 10^5 \text{ km/s} \) independent of the inertial system.

The second postulate leads to a very different forms transformation laws between inertial systems, mixing space and time (the so-called Lorentz transformation).
\[ x' = \gamma (x - vt), \quad ct' = \gamma (ct - \beta x) \quad \text{with} \quad \beta = \frac{v}{c}, \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}} \quad (8) \]

Note that a) for \( v \ll c \) this gives \( x' \approx x - vt \) and \( t' \approx t \) which are the Galilean transformations of Newtonian dynamics.

b) if we see light propagating in a system S: \( x = ct \rightarrow \) from there we obtain \( x' = ct' \), so it should be from the second postulate.

Once space and time are mixed, to write physical laws we need 4-vectors, because according to i) they must be valid in any inertial frame (IF), and we go from one IF to another by Lorentz transformations. (This is for the same reasons we needed vectors in classical physics, to write equations independent of coordinate systems.) Now, coordinates are needed for spacetime, the most basic 4-vector denotes an event, i.e. space & time coordinates:

\[ x^\mu = (t, x, y, z) \quad \text{or} \quad x^\mu = (t, \vec{x}) \]

and distance in spacetime is the interval \( ds^2 \):

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -c^2 dt^2 + dx^2 + dy^2 + dz^2 \]

which is invariant under Lorentz transformations (such in the same way as distance in 3D is inv. under coordinate transformations) - It is easy to see that \( ds = 0 \) must be conserved since this is a direct consequence of the speed of light being invariant. A pair of events have interval \( ds^2 < 0 \) are said to be spacelike (you can find an IF where \( dt' = 0 \)) and those with \( ds^2 > 0 \) are said to be timelike (you can find an IF where \( d\vec{x} = 0 \)). All four vectors transform from one IF to another by Lorentz transformations. Another important 4-vector is the 4-velocity:

\[ \mathbf{u}^\mu = \frac{dx^\mu}{d\tau} = \left( \frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau} \right) = \gamma \left( 1, \vec{u} \right) \quad (c \approx 1) \]

\( \tau \) proper time: time measured by observer at rest w/ particle.
\[ u^\mu u_\mu = \frac{1}{c^2} \left[ -1 + v^2 \right] = -1 \]

In SR we define the 4-momentum as

\[ p^\mu = m u^\mu \]

where \( m \) is rest mass of particle.

The components are

\[
\begin{cases}
  p^0 = E = mc^2 \gamma &= m c \gamma \\
p^\perp &= m \gamma \vec{v} \\
(p^\mu)(p^\nu) &= -m^2
\end{cases}
\]

The starting point for GR is the equivalence principle, that is, that gravity is locally equivalent to uniform acceleration. This is a well known fact that follows from the equality of inertial and gravitational masses \( m_I = m_G \). That is,

\[ F = m_I a = \frac{GMmG}{r^2} \Rightarrow a = \frac{GM}{r^2} \]

independent of mass of test particle as long as \( m_I = m_G \).

This gives the unusual property (when compared to other interactions, e.g. electromagnetic) that any object in the gravitational field of the earth will move in the same trajectory for fixed initial conditions (initial position and velocity) independent of its mass. For example, in electric case trajectories also depend on \( q/m \). The equality \( m_I = m_G \) is extremely well tested to an accuracy of \( 1.5 \times 10^{-13} \) (by looking at how the earth & moon fall into the Sun).

Due to this universality, gravity is locally equivalent to uniform acceleration:

For this observer, he sees the same behavior of the two masses or in Earth.
The equivalence is only local, because over large distances one can detect differences if the gravitational field is not uniform. Remarkably, this simple analogy immediately tells us that a light ray will be deflected in a gravitational field (because that's what you'll see in "Einstein's elevator"):

![Diagram of light ray in a gravitational field]

This universality of trajectories led Einstein to think that gravity may not be a force at all, but rather a property of spacetime. In GR spacetime is described by a metric $g_{\mu\nu}$ that becomes a dynamical variable, and it responds to the energy-momentum content of sources (this response is encoded in Einstein's equations, the equivalent of Poisson's equation in Newtonian gravity, $\nabla^2 \Phi = 4\pi G \rho$).

We will just state that in the case of a source of mass $M$ at the origin, the metric is the so-called Schwarzschild solution:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{tt} dt^2 + g_{rr} dr^2 + r^2 d\Omega^2$$

with $g_{tt} = -\frac{1}{g_{rr}} = -\left(1 - \frac{2GM}{c^2 r}\right) = -\left(1 + \frac{\Phi}{c^2}\right)$

where $\Phi$ is the Newtonian gravitational potential of the source, and due to spherical symmetry $g_{tt}$ and $g_{rr}$ depend only on $r$ (also note that the metric is static, the "Birkhoff theorem" guarantees that).

Due to the symmetries of this metric, there will be conserved quantities, same as for the Kepler problem in Newtonian case. The basic idea in GR is that gravity is due to free motion in curved spacetime with metric $g_{\mu\nu}$. Free motion is given by geodesics, which in flat space ("lemons") correspond to straight lines.
Geodesics are the generalizations of straight lines to curved spaces. Consider the following definition of a straight line:

"The tangent to such line at one point is parallel to the tangent at previous point."

What do we mean by parallel? Well, we parallel transport the tangent vector $\mathbf{v}_A$ and compare it to $\mathbf{v}_B$ at new point $B$ from $A$ to $B$.

Note that, in general, we can parallel transport any vector $\mathbf{v}$ along a curve defined by tangent vector $\mathbf{v}^\alpha$:

Parallel transport means that the derivative of the vector $\mathbf{v}^\alpha$ does not change along the curve, in other words, the directional derivative of $\mathbf{v}^\alpha$ along $\mathbf{v}$ is zero:

$$(\mathbf{v} \cdot \nabla) \mathbf{v} = 0 \quad \text{(parallel transport)}$$

A geodesic is a curve defined by a tangent vector $\mathbf{v}^\alpha$ that is parallel transported along itself:

$$(\mathbf{v} \cdot \mathbf{v}) \mathbf{v} = 0$$

In $\text{SR}$, we generalized this to 4-vectors:

$$\nabla^\alpha \mathbf{u}_\beta = 0 \quad \text{algebra: } \frac{\partial \mathbf{u}_\beta}{\partial x^\gamma} = \frac{\partial \mathbf{u}_\beta}{\partial x^\gamma}$$
In GR, the principle of equivalence says that any physical law that is expressed in tensor notation in SR is valid in a local inertial frame in GR. That means that the parallel transport condition reads in GR:

\[ U_\beta \; U^\alpha, \beta = 0 \]

where \( U_\beta \) denotes the covariant derivative, which generalizes parallel transport to curved spacetime:

\[
\frac{dU^\alpha}{d\lambda} = U^\alpha_{\;\beta} \frac{dx^\beta}{d\lambda} + \Gamma^\alpha_{\beta\gamma} U^\beta U^\gamma
\]

Now, in our case the geodesic is described by the motion of our test particle, i.e., its 4-velocity or 4-momentum which is tangent to the geodesic, therefore:

\[ p^\beta p_\beta; \alpha = 0 \implies \frac{d}{d\tau} \left( \begin{array}{c} m \frac{dp^\alpha}{d\tau} \\ \frac{1}{2} g_{\alpha\beta} p^\alpha p^\beta \end{array} \right) = \text{algebra} \]

Since \( \tau \) is proper time along curve:

\[ p^\alpha p_\alpha; \alpha = 0 \]

Now we have the important result that if all components of metric are independent of some \( x^\rho \), then \( p_\beta \) is a constant of motion along particle trajectories.

Let's go back to the Schwarzschild metric. Since \( g_{\mu\nu} \) is time independent, then \( p_0 \) is conserved (energy integral),

\[ p_0 = -Em \text{ (where } E \text{ is energy per unit mass)} \]

Similarly, independence of metric with respect to \( \phi \) rotations means \( p_\phi \) is conserved (angular momentum integral).
Again, define:

\[ p_\theta = m \frac{h}{r} \]  
(\text{where } h \text{ is angular momentum per unit mass})

Because of spherical symmetry, motion is confined to a single plane, which we take to be the equatorial plane \((\theta = \pi/2)\), this means \( p^\phi = m \frac{d\phi}{dt} = 0 \). The components of 4-momentum are:

\[
\begin{align*}
p^0 &= g^{00} p_0 = g^{00} p_0 = \frac{mE}{1 + 2\Phi} \\
p^r &= m \frac{dr}{dt} \\
p^\theta &= g^{\theta\theta} p_\theta = \frac{m h}{r^2} \\
p^\phi &= g^{\phi\phi} p_\phi = 0
\end{align*}
\]

\[
\text{Note:} \quad g^{\mu\nu} v_\mu v_\nu = \delta \mu \nu
\]

\[
\begin{align*}
g^{\mu\nu} &= \delta^{\mu\nu} \\
g^{00} &= 1 \\
g^{rr} &= \frac{1}{g_{rr}} \\
g^{\theta\theta} &= \frac{1}{g_{\theta\theta}}
\end{align*}
\]

Now, we have the simple relation \( p^\mu p_\mu = -m^2 \) (again, since valid in tensorial form in SR, carries over to potential frames in GR) (since no derivatives are involved, no modification needed)

\[
\Rightarrow - \frac{m^2 E^2}{1 + 2\Phi} + \frac{m^2}{1 + 2\Phi} \left( \frac{dr}{dt} \right)^2 + \frac{m^2 h^2}{r^2} = - m^2
\]

\[
\Rightarrow \left( \frac{dr}{dt} \right)^2 = E^2 - (1 + 2\Phi) \left( 1 + \frac{h^2}{r^2} \right) = E^2 - V_{eff}
\]

where the effective potential is given by,

\[
V_{eff} = \sqrt{(1 + 2\Phi) \left( 1 + \frac{h^2}{r^2} \right)} \quad \text{large } r \rightarrow 1 + \Phi + \frac{h^2}{2r^2}
\]

which, up to a constant, is the usual Newtonian effective potential (always per unit mass). We can examine orbits by using this effective potential much in the same way we did in Newtonian case.
There are striking differences for small radius between the two potentials, but even the circular orbits are different. In Newtonian case we have

\[ \frac{\partial V_{\text{eff}}}{\partial r} = 0 = \frac{GM}{r^2} - \frac{h^2}{r^3} \Rightarrow r_{\text{circ}} = \frac{h^2}{2GM} = \frac{h^2}{\mu} \]

In GR case \( \frac{\partial V_{\text{eff}}}{\partial r} = 0 \) gives

\[ r_{\text{circ}} = \frac{h^2}{2MG} \left( 1 \pm \sqrt{1 - \frac{12M^2G^2}{h^2}} \right) \]

For only if \( h^2 > 12M^2G^2 \) there are circular orbits; and at \( h^2 > 12M^2G^2 \)

So, only if \( h^2 > 12M^2G^2 \) there are circular orbits; and at \( h^2 < 12M^2G^2 \)

there is a second (unstable) circular orbit. Note that for large specific angular momentum the GR stable circular orbit becomes equal to Newtonian one.

Now, let's look at the equation for the orbit. We have

\[ \frac{d\phi}{dt} = \frac{p_\phi}{m} = g \phi \frac{p_\phi}{m} = \frac{h}{r^2} \]

\[ \Rightarrow \left( \frac{dr}{d\phi} \right)^2 = \left( \frac{dr}{dt} \right)^2 \left( \frac{d\phi}{dt} \right)^2 = \frac{r^4}{h^2} \left( E^2 - V_{\text{eff}} \right) \]

Now, we do some thing as before, let \( u = \frac{1}{r} \), then:

\[ \text{as we approach Schwarzschild radius } 2GM/c^2 \text{, but we will not get there for solar system } 2GM = 3 \text{ km, whereas } R_0 = 2 \times 10^5 \text{ km, Sun is not a blackhole!} \]

from previous class
\[ \frac{d\phi}{d\phi} = \frac{d}{d\phi}(u) = -\frac{1}{u^2} \frac{du}{d\phi} \]

\[ \Rightarrow \left(\frac{du}{d\phi}\right)^2 = \frac{E^2 - (1+2\varepsilon)(1+\varepsilon^2 u^2)}{\hbar^2} \]

Then, taking \(d/d\phi\) of this we get:

\[ 2 \frac{du}{d\phi} \frac{d^2u}{d\phi^2} = \frac{2GM}{\hbar^2} \frac{du}{d\phi} - 2u \frac{du}{d\phi} + 6GM \frac{u^2}{c^2} \frac{dM}{d\phi} \]

\[ \Rightarrow \frac{d^2u}{d\phi^2} + u = \frac{6GM}{\hbar^2} + 3GM \frac{u^2}{c^2} \]

That is the full equation for the orbit in GR, note it has an additional term in the RHS.

To solve this non-linear ODE, we do perturbation theory about a Newtonian circular orbit, i.e. assuming eccentricity \(\varepsilon\) is a small parameter. Remember that comoving solution in Newtonian

\[ y_{\text{Newt}} = \frac{(6GM)}{\hbar^2} (1 + \varepsilon \cos \phi) \quad \text{(setting } \omega = 0) \]

Now, let \( y = \frac{y_{\text{Newt}}}{y_{\text{circ}}} - 1 \)

\[ y_{\text{Newt}} = e \cos \phi \]

\[ \Rightarrow \frac{d^2y}{d\phi^2} + y = \frac{3(6GM)^2}{\hbar^2} (y+1)^2 = \frac{3(6GM)^2}{\hbar^2} (1 + 2y + y^2) \]

Now, see what happens: first term in RHS corrects for circular orbit radius, the second term is the one we are after (the third we neglect because we assume \(\gamma \approx 1\)).

It gives a correction to the frequency of the oscillator.
\[ y \rightarrow y \left( 1 - 6 \frac{Gm^2}{h^2} \right) = k^2 y \]

\[ \Rightarrow \text{solution in GR will be} \quad \frac{dy}{d\phi^2} + k^2 y = g_0 \Theta(y) \propto y_0 \]

\[ \Rightarrow y = y_0 + B e^{k\phi} \cos \left[ k(\phi - \omega) \right] \quad y_0 = \frac{3}{h^2} \frac{G^2 m^2}{\mu^2} \]

Since \( k = 1 \) it means that orbit returns to same radius when \( \phi \) goes through \( \pm 2\pi \), then the change from one perihelion to the next is given by

\[ \Delta \phi = \frac{2\pi}{k} = 2\pi \left[ 1 - 6 \frac{G^2 m^2}{h^2} \right]^{-1/2} \approx 2\pi \left( 1 + 3 \frac{G^2 m^2}{h^2} \right) \]

Thus, the perihelion shift is:

\[ \Delta \phi_{\text{shift}} = 6\pi \frac{G^2 m^2}{h^2} \quad \text{radians per orbit} \]

To evaluate for Mercury, use angular momentum from circular orbit:

\[ h^2 = \frac{GM \text{R}_{\text{circ}}}{1 - 3 \frac{Gm}{c^2}} \approx \frac{GM \text{R}_{\text{circ}}}{1} \quad \text{non-relativistic} \]

\[ \Rightarrow \Delta \phi_{\text{shift}} = 6\pi \frac{GM}{\text{R}_{\text{circ}}} \approx 3\pi \frac{R_s}{\text{R}_{\text{circ}}} \]

For Mercury, \( \text{R}_{\text{circ}} = 5.55 \times 10^7 \text{ km} \)

\[ T_{\text{Mercury}} = 0.24 \text{ yr} \]

\[ \Rightarrow \Delta \phi_{\text{shift, Mercury}} = 5 \times 10^{-7} \text{ radians per orbit} \]

\[ = 0.43'' \text{ / yr} = 43'' \text{ / century} \]

\[ R_s = \frac{2GM}{c^2} \]
The experimental values (corrected for Newtonian precession) compared to observed values are:

<table>
<thead>
<tr>
<th>Planet</th>
<th>GR</th>
<th>OBS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mercury</td>
<td>43</td>
<td>43.11 ± 0.45</td>
</tr>
<tr>
<td>Venus</td>
<td>8.6</td>
<td>8.4 ± 4.8</td>
</tr>
<tr>
<td>Earth</td>
<td>3.9</td>
<td>5.0 ± 1.2</td>
</tr>
<tr>
<td>Mars</td>
<td>10.3</td>
<td>9.8 ± 0.8</td>
</tr>
</tbody>
</table>

Increase as \( v \to \infty \) predicted by GR.