Homework Set #4 (Due Friday, December 14th)

1) Consider the Poisson equation,
\[ \nabla^2 \Phi = 1 \]  
(1)
in an irregular two-dimensional geometry defined by the equations,
\[ 0 \leq x \leq 1.5, \quad 0 \leq y \leq 2, \quad y \geq 1.5 - 2x, \quad y \leq 2.75 - 1.5x \]  
(2)
with boundary conditions $\Phi = 0$.

a) Write a code that solves this using Gauss-Seidel relaxation in a cartesian grid.

b) Solve the problem for a grid of size $100 \times 100$. Plot the error at step $n$ as a function of $n$ defined as
\[ \text{error}(n) \equiv \text{Max } |\Phi^{(n)}(x, y) - \Phi^{(n-1)}(x, y)|, \]  
(3)
from $n = 1$ until the solution converged to a reasonably small error.

c) Make a contour plot of $\Phi(x, y)$ for your solution.

d) Repeat a)-c) for the SOR method (choose the parameter $1 < w < 2$). Compare the convergence rate against Gauss-Seidel.

2) Consider the following linear PDE (with $a$, $b$ constants and $n$ integer),
\[ \frac{\partial f}{\partial t} + b x \frac{\partial f}{\partial x} = a x^n. \]  
(4)

a) Solve Eq. (4) by the method of characteristics. Express your solution in terms of the initial condition $f(x, t = 0) = f_0(x)$.

b) Plot your solution as a function of $x$ at different times $t = 0.1, 2$, for $f_0(x) = \exp(-(x - 1)^2)$, $b = n = 1$. Consider two cases, $a = 0$ and $a = 0.05$. Explain what you see in the plots.
3) Consider the Ising model in a square lattice in two dimensions of size $L = 50$ with free boundary conditions. Normalize the maximum magnetization to unity (i.e. all spins up).

a) Use the Metropolis algorithm to find the equilibrium state at a given temperature, choosing the number of Monte Carlo steps per spin $t_n = n/L^2$ such that a “cold” initial condition (e.g. all spins up) and a “hot” initial condition (all spins random) gives approximately the same magnetization after $n$ flip proposals.

b) Calculate the magnetization as a function of temperature and make a plot of it, comparing with the analytic prediction of the critical temperature $k_B T_c = 2.269 J$. At each temperature calculate the magnetization by averaging over 5 seeds of initial “hot” conditions, each of them evolved through $t_n$ steps of the chain.