1) In general relativity (GR), the orbit of a test particle around a source obeys the following differential equation (see lecture notes),

\[
\frac{d^2 u}{d\phi^2} + u = \frac{GM}{h^2} + 3 \frac{GM}{c^2} u^2, \tag{1}
\]

where \( u \equiv 1/r \) is the inverse radial position of the orbit, \( M \) is the source mass, \( h \) is the specific angular momentum of the orbit, \( G \) is Newton’s constant, and \( c \) is the speed of light. Compared to the Newtonian result, Eq. (1) contains a relativistic correction (depending on \( c \)) that is of order \((GM/c^2 r)\) of the Newtonian, linear in \( u \), term. This dimensionless number is characteristic of GR, since \((GM/c^2 r)\) is roughly the ratio of the Schwarzschild radius of the source \((2GM/c^2)\), where GR gravity is very different from Newton’s) to the characteristic size of the orbit \( r \).

a) Write a code that, using 4th-order explicit Runge-Kutta, solves Eq. (1).

b) Use your code for the specific case of Mercury orbiting around the Sun\(^1\). Switching off the relativistic term in Eq. (1), check that your code produces a closed orbit and compare it to the exact Newtonian result,

\[
u = \frac{GM}{h^2} \left( 1 + e \cos(\phi) \right), \tag{2}
\]

where the eccentricity obeys \( h^2/GM = a(1 - e^2) \) with \( a \) the semi-major axis of the orbit. Make a plot of this comparison for a few different choices of the timestep in your integrator. For initial conditions, you can start for example at perihelion where \( du/d\phi = 0 \).

c) Use your code for the relativistic case and compare the result you get for the perihelion precession with that of the perturbative result we obtained in class. **Hint:** at the beginning, you may want to artificially increase the precession to make easier its detection by amplifying arbitrarily the last term in Eq. (1). Also, in order to avoid making the timestep very small (which makes it slow) to find the perihelion, you may want to fit a quadratic to the three points closest to perihelion to better find the angle at perihelion.

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\(^1\)For Mercury parameters see e.g. nssdc.gsfc.nasa.gov/planetary/factsheet/mercuryfact.html. In your writeup, please state clearly the information you use.
2) Suppose the gravitational Newtonian force is changed to

$$ F = \frac{GMm}{r^2} \times \left( \frac{r_0}{r} \right)^\delta $$

(3)

a) Find the equation of the orbit for this force law.

b) Using your Runge-Kutta solver, compute the perihelion shift for $\delta = 0.05$ and $r_0 = \frac{h^2}{GM}$, where $h$ is the angular momentum per unit mass.