Recall that in Newtonian core we had,

\[ \dot{\delta p} = -3H \delta p - 3 \frac{\rho}{\bar{p}} \delta \eta \]  

(Continuity)

\[ \dot{\delta \eta} = -2H \delta \eta - \frac{4\pi G}{3} \delta \rho - \frac{1}{3} \frac{\nabla_r^2 \delta \rho}{\bar{p}} \]  

(Euler)

where \( \delta \eta = \frac{\nabla_r \cdot \bar{\nabla}}{3} = \frac{1}{a} \frac{1}{3} \nabla_r \bar{\nabla} \equiv \frac{\Theta}{3a} \)

For the assumed MAT dominated universe, let's assume that \( \bar{p} \propto a^{-3} \) \((\text{w = -1})\), \( \Theta = \frac{4\pi G}{3} \rho \text{ MAT} \) \((-1 \text{ vacuum})\).

Here \( w = \bar{p}/\bar{\rho} \) defines equation of state

\[ \Rightarrow \dot{\delta p} = -3(\eta w) \frac{\delta \eta}{\bar{p}} \]

In a relativistic core the equations of motion are the same (in the so-called "Newtonian" conformal gauge) if we just replace \( \bar{p} \rightarrow \bar{p} + \bar{\rho} \) and the Newtonian equations!

Then,

\[ \begin{cases} \dot{\delta p} = -3H \delta p - 3(\bar{p} + \bar{\rho}) \delta \eta \\ \dot{\delta \eta} = -2H \delta \eta - \frac{4\pi G \delta \rho}{3} - \frac{1}{3} \frac{\nabla_r^2 \delta \rho}{\bar{p} + \bar{\rho}} \end{cases} \]

This is a lame way of doing this, sorry we don't have time for deriving this from GR, see 4.3.

in Padmanabhan.

as well \( \delta \rho = \bar{p} \delta \)

Continuity reads

\[ -3(\eta w) \frac{\delta \eta}{\bar{p}} \delta p + \bar{p} \frac{\partial \delta p}{\partial t} = -3H \frac{\delta \rho}{\bar{p}} \delta \frac{\Theta}{\bar{p}} - (\frac{\bar{p} + \bar{\rho}}{a}) \]

\[ \Rightarrow \frac{\partial \delta p}{\partial t} = \eta w (\bar{p} - \bar{\rho}) + 3 \frac{\Theta}{\bar{p}} \bar{\rho} \delta \]

\[ \Rightarrow \frac{\partial \delta \rho}{\partial t} = -\eta w (\bar{p} - \bar{\rho}) + 3 \frac{\Theta}{\bar{p}} \bar{\rho} \delta \]

**Note:** The equations and notation used in this document are those commonly encountered in general relativity and cosmology. The specific context and assumptions are important to fully understand the implications of these equations.
The equation is,

\[-\frac{H}{a} \dot{\theta} + \frac{1}{a} \theta = -\frac{2}{3} \theta \frac{H'}{H} - 4\pi G \rho \delta - \frac{1}{a^2} \frac{\nabla^2 \delta \rho}{\rho}\]

\[\Rightarrow \delta_{\theta} + H \theta \delta_{k} = -\frac{3}{2} \theta^2 \delta_{k} + \frac{H^2}{1+\omega} \delta_{k} \]

We for simplicity we assume \(\omega = 1\).

Now, let's find second order diff equation for \(\delta_k\):

\[\frac{\partial^2 \delta}{\partial t^2} = -\omega \theta - (1+\omega) \left[ -\frac{H}{H'} + \frac{3}{2} H^2 \delta + \frac{H^2}{1+\omega} \delta \right] + 3 \omega H \delta + 3 \omega \frac{\delta}{t} + 3 \omega H \frac{\partial \delta}{\partial t} \]

where \(\frac{\delta}{t} = \frac{\partial}{\partial t} - \frac{3}{2} \frac{\partial}{\partial \bar{t}} \quad \text{(Recall that,)}\)

\[\begin{cases} \dot{H} &= -\frac{3}{2} H^2 (1+\omega) + H \delta \\ \omega' &= 3 \omega H (\omega - c_s^2) (1+\omega) \end{cases} \quad \text{(Friedmann equations)} \]

\[\text{From } \bar{\rho} = \omega \bar{\rho} \quad \text{and } \bar{\rho} = (\bar{s}^2 \bar{\rho}) \]

Then, we have:

\[\frac{\delta_{\delta_k}}{\partial t^2} = -3 H (\omega - c_s^2) \frac{H}{1+\omega} \left[ 3 \omega H \delta - \frac{\delta}{\bar{t}} \right] + \frac{H}{1+\omega} H \left[ 3 \omega H \delta - \frac{\delta}{\bar{t}} \right] \]

\[+ (1+\omega) \frac{3}{2} H^2 \delta - c_s^2 \delta + 3 \omega H (\omega - c_s^2) (1+\omega) \delta + 3 \omega H^2 \left[ 1 - \frac{3}{2} (1+\omega) \right] \delta + 3 \omega H \frac{\delta}{\bar{t}} \]

\[= H^2 \delta \left[ 6 \omega + \frac{3}{2} (1+\omega) (1-3\omega) \right] - c_s^2 \delta + \frac{H \delta}{\bar{t}} \left[ 6 \omega - 1 - 3 c_s^2 \right] \]

\[\text{Since we are assuming } \bar{\omega} = 1, \text{ solutions will depend on time only through } \bar{t}, \text{ so it is convenient to use } \bar{t} \text{ as time.} \]
\[
\frac{1}{H} \frac{\partial H}{\partial t} = \frac{3}{\Omega_m} + \frac{1 - \frac{3}{2} \left(1 + \omega \right)}{\frac{H^2}{\Omega_m}} + \left(\frac{1}{\Omega_m} \frac{\partial \Omega_m}{\partial t}\right) \frac{2}{\Omega_m}
\]

Here:

\[
\frac{\partial^2 \delta}{\partial \Omega_m^2} + \left[1 - \frac{3}{2} \left(1 + \omega \right)\right] \frac{\partial \delta}{\partial \Omega_m} = \left(6\omega - 1 - 3\omega^2\right) \frac{\partial \delta}{\partial \Omega_m} - \frac{h^2 c_s^4}{H^2} \delta
\]

\[
+ \delta \left[9\omega - \frac{3}{2} - \frac{9}{2} \omega^2\right]
\]

For matter: \(c_s^2 = \omega = 0\), and \(\frac{k^2}{H^2} \ll 1\)

\[
\frac{\partial^2 \delta}{\partial \Omega_m^2} + \frac{1}{2} \frac{\partial \delta}{\partial \Omega_m} - \frac{3}{2} \delta = 0
\]

Try \(p^2 + \frac{p^2}{2} - \frac{3}{2} = 0\) \(\Rightarrow p = \pm 1, -3/2\)

The usual growing and decaying modes (also for scales larger than Hubble radius!)

For radiation: \(c_s^2 = \omega = 1/3\)

\[
\frac{1}{2} - \frac{15\omega}{2} + 3c_s^2 = \frac{1}{2} - \frac{5}{2} + \frac{1}{3} = -1
\]

\[
\frac{9}{2} \omega^2 - 3\omega - \frac{3}{2} = \frac{1}{2} - 1 - \frac{3}{2} = -2
\]
Then:
\[
\frac{\delta^2}{\partial a^2} - \frac{\delta}{\partial a} - 2\delta + \frac{k^2}{3H^2}\delta = 0
\]

\[
\Rightarrow \frac{\delta}{\partial \ln a} - \frac{\delta}{\partial a} = \left(2 - \frac{k^2}{3H^2}\right)\delta
\]

So if \( \frac{k^2}{3H^2} \ll 1 \),

\[
\Rightarrow \frac{\delta^2}{\partial a^2} - \frac{\delta}{\partial a} - 2\delta = 0 \Rightarrow \delta \propto a^p
\]

\[
\Rightarrow p^2 - p - 2 = 0 \Rightarrow [p = 2, -1]
\]

So perturbations in \( \text{RAD} \), for \( \frac{k}{aH} \ll 1 \), behave as

\[
\delta_k = A_k a^2 + B k a^{-1}
\]

However, in the opposite limit, where \( \frac{k}{aH} \) is large,

\[
\frac{\delta}{\partial a} - \frac{\delta}{\partial a} = a^2 \frac{\delta^2}{\partial a^2} = -\frac{k^2}{3H^2}\delta
\]

\[
\Rightarrow \frac{\delta^2}{\partial a^2} = -\frac{k^2}{3H^2 a^4}\delta
\]

Now, in \( \text{RAD} \) \( H^2 \sim \frac{\rho}{M} a^{-4} \Rightarrow H^2 a^4 = \text{const.} \)

\[
\Rightarrow \delta \propto \exp \left[ \frac{k^2 a^2}{\sqrt{3} H a^4} \right] \frac{1}{aH}\delta
\]

Again, sound waves.
Evolution of CDM during radiation era.

We now consider the evolution of fluctuations in dark matter when it becomes smaller than Hubble radius during RSD. When \( x < H^{-1} \), radiation is smooth (recall that \( c < H \) at high redshift for RSD is \( x^{-1} \)); but during RSD era \( \rho \) is dominated by dark matter, so it either dominates the evolution of Hubble constant.

We have:

\[
\frac{\delta^2 \delta}{\delta T^2} + \frac{H}{\delta T} \frac{\delta T}{\delta T} = 4 \pi G a^2 \delta \Phi_{\text{tot}} = 4 \pi G a^2 \left[ \frac{\bar{\rho}_M}{\bar{\rho}_R} \delta + \bar{\rho}_R \frac{\delta}{\delta T} \right]
\]

which now Friedmann equation is:

\[
H^2 = \frac{8 \pi G a^2}{3} \left( \frac{\bar{\rho}_R + \bar{\rho}_M}{\bar{\rho}_M} \right)
\]

It is convenient to work with time variable \( \tau = a/\rho_{\text{crit}} \)

\[
x \frac{d}{dx} = \frac{d}{d\tau} = \frac{a}{\rho_{\text{crit}}} \frac{dx}{d\tau} = \frac{a}{\rho_{\text{crit}}^2} \frac{dx}{d\tau} = \frac{a}{\rho_{\text{crit}}^2} \frac{dx}{d\tau}
\]

\[
H \frac{\delta}{\delta T} = H^2 x \frac{\delta}{\delta x}
\]

\[
H^2 = \frac{8 \pi G a^2}{3} \frac{\bar{\rho}_M}{\bar{\rho}_M} \left( 1 + \frac{1}{x} \right) \Rightarrow 4 \pi G a^2 \frac{\bar{\rho}_M}{\bar{\rho}_M} = \frac{1}{2} H^2 \frac{1}{1+x}
\]

\[
\frac{\delta^2 \delta}{\delta x^2} = H^2 x^2 \left( \frac{\delta}{\delta x} \frac{\delta}{\delta x} \right) = H^2 x^2 \frac{\delta^2 \delta}{\delta x^2} + H^2 x \frac{\delta}{\delta x} \frac{\delta}{\delta x} + H x^2 \frac{\delta}{\delta x} \frac{\delta}{\delta x}
\]

We need to get \( \frac{\delta H}{\delta x} \) from Friedmann equation.
\[ 2H \, dH = 4 \pi G \, \frac{8 \pi \rho_m}{3} \left[ \frac{2}{X} \left( 1 + \frac{1}{X} \right) - \frac{2}{X} \left( 1 + \frac{1}{X} \right) - \frac{1}{X^2} \right] \]

\[ \frac{d\rho_m}{dX} = -3 \frac{\rho_m}{X} \frac{X^2}{1+X} \]

\[ \Rightarrow \quad \frac{dH}{dX} = \frac{H^2}{1+X} - \frac{1}{2H} \left[ \frac{2}{X} \left( 1 + \frac{1}{X} \right) - \frac{1}{X^2} \right] = -\frac{3}{2} \frac{X+2}{X(X+1)} \]

Here, we have:

\[ \frac{d^2 \delta}{dX^2} + 2 \frac{d}{dX} \frac{X}{X+1} \frac{d \delta}{dX} - \frac{2H^2}{2} \frac{X^2 (X+2)}{X(X+1)} \frac{d \delta}{dX} = \frac{3}{2} \frac{X^2}{1+X} \delta \]

\[ 2X - X (X+2) = \frac{X (3X+2)}{2 (X+1)} \]

\[ \Rightarrow 2X (X+1) \frac{d^2 \delta}{dX^2} + (3X+2) \frac{d \delta}{dX} = 3 \delta \]

The solution can be guessed by setting \( \frac{d^2 \delta}{dX^2} = 0 \) \( \Rightarrow \delta = A \delta (3X+2) \)

We can see that this becomes the usual growing mode in MAT era.

For \( X > 1 \), \( 8 \pi a \). In the RAD era, however, \( X \ll 1 \) and growth is suppressed, this is because extra contribution to \( H \) speeds up the expansion of the universe leading to slower growth.

Thus, even though \( X > 1 \), the dark matter perturbations do not grow significantly during RAD era (for scales \( X < 1/H \)).

Remember for \( X > 1/H \), \( 8 \pi a \) during RAD era. To do the matching at \( X = H^{-1} \), we need to denote the decay up mode.
This is found by taking \( \delta = (3x+2)f \) and solving for \( f \):

\[
\frac{\delta_k}{\delta_0} = B_k \left[ (3x+2) \ln \left( \frac{\sqrt{1+x} + 1}{\sqrt{1+x} - 1} \right) - 6\sqrt{x+1} \right]
\]

which gives:

\[
\delta_k = \begin{cases} 
B_k \left( 2 \ln \frac{y}{Xe} - 6 \right) & \text{for } x \ll 1 \\
\frac{8B_k}{15} \frac{1}{x^{3/2}} & \text{for } x \gg 1
\end{cases}
\]

The latter is what we expect, \( \delta \propto a^{-3/2} \) during the MMT era.

How much does a mode that enters well in the RAP era \((x_{\text{enter}} = Xe \ll 1)\) grow until \( x = 1 \)? Since the mode has been super Hubble for a long time, we assume that by the time it enters it is in the growing mode for \( x > H^{-1} \), i.e.

\[
\delta(x) \equiv \frac{x^2}{Xe^2} \quad \text{for } x \ll Xe \quad \text{when } x > H^{-1}
\]

\( x \) is the same as for RAP perturbations, we are here assuming adiabatic (or curvature) perturbations; more on this later. However, this mode will not correspond merely to a growing mode \((3x+2)\) by the time it becomes sub-Hubble, it will be a linear combination of growing and decaying modes. To find that combination we impose matching conditions:

At \( x = Xe \):

\[
\begin{align*}
Xe &= 2A - 6B + 2B \ln \frac{y}{Xe} \\
2Xe &= 3A - 2B \frac{1}{Xe}
\end{align*}
\]

\( \Rightarrow \)

\[
B = \frac{Xe}{2}
\]

\[
\begin{align*}
\frac{\delta}{\delta x} \text{ outside} &= \frac{\delta}{\delta x} \text{ inside} \\
\frac{\partial \delta}{\partial x} \text{ outside} &= \frac{\partial \delta}{\partial x} \text{ inside}
\end{align*}
\]
\[ x_e^2 - 6x_e = 2A - 2x_e \ln \left( \frac{4}{x_e} \right) \]

\[ A = \frac{x_e^2}{2} \left[ 2\ln \left( \frac{4}{x_e} \right) - 5 \right] \]

(Ans 3b(x_e) so throwing away 3A), was OK to solve for B.

Since it crosses, we have conversion into growing and decaying mode:

\[ \delta(x_e) = x_e^2 \left[ \ln \left( \frac{4}{x_e} \right) - \frac{5}{2} \right] (3x_2 + 2) - x_e^2 \left[ (3x_2 + 2) \ln \left( \frac{1 + x}{\sqrt{x + 1} - 1} \right) - 6 \sqrt{1 + x} \right] \]

Note the negative sign.

\( x_e < 1 \)

By the time the universe becomes MAT dominated, \( a = a_{BBN} \Rightarrow x = 1 \).

We have an amplification:

\[ \frac{\delta(x = 1)}{\delta(x_e)} = \frac{5x_e^2 \ln(x_e^{-1})}{x_e^2} = \frac{5 \ln \left( \frac{a_{BBN}}{\text{center}} \right)}{x_e^2} \]

This is only logarithmic! (This only holds for \( x_e < 1 \).)

Note that the amplification factor depends on the wavelength of the mode (through \( a_{BBN} \)), so different modes will be amplified slightly differently, leading to a change in the power spectrum.

Summarizing the evolution of perturbations:

- \( H^2 < 1 \): WDM era, \( \lambda > H^{-1} \)
  - \( 1 \): \( \lambda < H^{-1} \), RAD era,\( \delta_R \)
  - \( 1 \): \( \lambda < H^{-1} \), before DEC, oscillates
  - \( 1 \): \( \lambda < H^{-1} \), after DEC, \( \delta_B \)

\( a^2 \) oscillates coupled to \( \delta_{BBN} \).

For \( \lambda > H^{-1} \), DM is not dominant.