Recall Hubble-Bondi equation:

\[ H^2 = \frac{8\pi G}{3} \quad \rho - \frac{\mathcal{K}}{a^2} \quad \text{(we take } \mathcal{K} \text{ instead of } k \text{, not to confuse with Fourier modes!)} \]

For the perturbed case, we define the curvature perturbation in an analogous way:

\[ \delta K = \frac{-\delta K}{a^2} \quad \text{(we assume } \mathcal{K} = 0) \]

\[ H^2(\mathbf{x}, t) = \frac{8\pi G}{3} \rho(\mathbf{x}, t) + \frac{2}{3a^2} \nabla^2 \delta K(\mathbf{x}, t) \]

This corresponds in Fourier space to:

\[ -\frac{2}{3} \frac{k^2}{a^2} \delta \delta K = -\frac{1}{6} \left( \frac{4 k^2}{a^2} \mathcal{K} \right) \]

In what follows we assume \( \mathcal{K} = 0 \).

We want to find how \( \delta \delta K \) evolves outside Hubble radius, to connect perturbations generated by inflation to how they come back inside Hubble radius later. Also, we seek the connection of \( \delta \delta K \) to \( \Phi \), the gravitational potential so we can translate fluctuations in \( \delta \delta K \) to density fluctuations. [Remind that \( \delta \delta K = -\left( \frac{H}{\rho} \delta \delta \Phi \right) \) from inflation]

To find evolution of \( \delta \delta K \), we recall the evolution of fluctuations in relativistic case:

\[
\begin{align*}
\delta \rho &= -3H \delta \rho - 3 \left( \rho + \dot{\rho} \right) \delta H \\
\dot{\delta} &= -2H \delta H - \frac{4\pi G}{3} \delta \rho - \frac{1}{3} \frac{\nabla^2 \delta \rho}{a^2 (\rho + \dot{\rho})} \\
\end{align*}
\]
To find evolution of $R_k$ we take time derivatives from
\[
\left(\frac{H^2}{\langle \rho \rangle}\right) = \frac{8\pi G}{3} \rho (x, t) + 2 \frac{\overline{a^2}}{3} \nabla^2 R(x, t)
\]

The homogeneous part will not involve $R$, obviously, so we only need to look at time derivative of above in first order perturbations:
\[
2H \delta H_k + 2 \delta H_k H = \frac{8\pi G}{3} \delta \rho - \frac{2}{3} \frac{a^2}{a^2} R_k \delta \rho + 4 \frac{\overline{a^2}}{3} \delta R_k H
\]
\[
\Rightarrow \quad R_k = \frac{3a^2}{2H^2} \left[ \frac{8\pi G}{3} \delta \rho - 2 \left( H \delta H_k + \delta H_k H \right) \right] + 2 \frac{\overline{a^2}}{3} \delta H
\]

Now, we use equations of motion for $\delta \rho$, $\delta H$, plus $H = -\frac{8\pi G}{3} \langle \rho \rangle$

\[
\Rightarrow \quad R_k = \frac{3a^2}{2H^2} \left[ \frac{8\pi G}{3} \left( -3H \delta \rho - 3 \langle \rho \rangle H \right) - 2H \left[ -2H \delta H - \frac{8\pi G}{3} \delta \rho - \frac{1}{3} \frac{\overline{a^2}}{a^2} \langle \rho \rangle H \right] \right]
\]
\[
- 2 \delta H \left[ -\frac{8\pi G}{3} \langle \rho \rangle \right]
\]
\[
= \frac{3a^2}{2H^2} \left[ \delta \rho \left[ \frac{8\pi G}{3} (-3H) + \frac{8\pi G}{3} H \right] + \delta H \left[ -\frac{8\pi G}{3} \langle \rho \rangle + 4H^2 + 8\pi G \langle \rho \rangle \right] \right]
\]
\[
+ \frac{3a^2}{2H^2} \frac{\overline{a^2}}{\langle \rho \rangle} \delta R_k \left[ -H \right] + 2 \frac{\overline{a^2}}{3} \delta H
\]
\[
\Rightarrow \quad R_k = -H \frac{\delta R_k}{\langle \rho \rangle} + \frac{3a^2}{2H^2} \left[ \frac{4\overline{a^2}}{3a^2} R_k H + 4H^2 \delta H - 2H \frac{8\pi G}{3} \delta \rho \right]
\]

From perturbing local Friedmann we have:
\[
2H \delta H = \frac{8\pi G}{3} \delta \rho - \frac{2}{3} \frac{a^2}{a^2} R_k \Rightarrow \frac{4\overline{a^2}}{3a^2} R_k H + 4H^2 \delta H - 2H \frac{8\pi G}{3} \delta \rho = 0
\]
\[ R_k \approx -H \frac{\delta \rho_k}{\delta + \Phi} \]

see how much \( R_k \) changes in a Hubble time, we look at,

\[ \frac{\dot{R}_k}{R_k} = \frac{1}{H} \frac{\partial \ln R_k}{\partial t} = -\frac{\delta \rho_k}{R_k} \frac{1}{\delta + \Phi} \approx \frac{2}{3} \frac{\delta \rho_k}{R_k} \frac{H^2}{\delta + \Phi} \frac{\delta \Phi}{\delta + \Phi} \frac{1}{R_k} \]

Poisson

\[ (-H \frac{\delta \Phi}{\delta + \Phi} = \frac{3}{2} H^2 \frac{\delta \rho_k}{\delta + \Phi}) \]

[technically, since \( R_k \) is gauge inv at z = 0 \( \frac{\dot{R}_k}{R_k} \) can eval. \( R \) in any gauge, Poisson is written as true in tot. matter gauge]

so, for \( \frac{1}{H} \frac{\partial \ln R_k}{\partial t} \) to be negligible outside the Hubble radius, \( (\frac{\delta \rho_k}{\delta + \Phi}) \ll 1 \)

all we need is to show that \( \frac{\delta \rho_k}{R_k} \) and \( \frac{\delta \Phi}{(\delta + \Phi) R_k} \) do not change significantly with \( z \)

at least one is of order unity.

For adiabatic fluctuations, which we are considering here, \( \frac{\delta \rho_k}{\delta + \Phi} \approx \frac{\delta \rho_k}{\delta + \Phi} \)

which is of order unity and does not depend - change significantly on super-Hubble evolution during times of interest.

Adiabatic fluctuations are so called because since they correspond to fluctuations on the local value of the local curvature, all species participate equally (by eigenframe rule),

\[ \frac{\delta n_i}{n_i} = \frac{\delta n_i}{n_i} = \frac{\delta n_i}{n_i} = \frac{\delta \rho}{\delta + \Phi} = \frac{\delta n}{\delta + \Phi} \]

(\( n \) : entropy density)

\[ \Rightarrow \delta (n_s) = \frac{\delta n}{\delta + \Phi} \Rightarrow n_s \Rightarrow \delta (n_s) = \frac{\delta n_i}{n_i} \Rightarrow \delta (n_s) = \frac{\delta n_i}{n_i} - \frac{\delta n}{\delta + \Phi} = 0 \]

\[ \Rightarrow \text{the fluctuation in the number per comoving volume vanishes} \Rightarrow \text{adiabatic} \]

(Also, for some reason \( \delta (n_s) < 0 \), entropy per matter particle is increased)

That is, if we expand or contract some volume element by some factor, all species experience the same change in number density.
There are also isocurvature fluctuations, in which the total energy density perturbation consists of, i.e. 2 components where $\delta \phi_1 + \delta\phi_2 = 0$. We shall not consider these perturbations (mostly because their contribution to the total energy density must be small compared to adiabatic fluctuations, from constraints coming from CMB fluctuations).

We now want to see that $\frac{\delta \Phi_k}{(\tau_w) R_k}$ is not large. We get such a result by solving at the continuity equation,

$$\delta \dot{\phi} = -3 (\delta + \overline{\delta}) \delta H - 3H \delta \phi$$

Poisson equation $-\kappa^2 \Delta \phi = \frac{3}{2} H^2 \delta H = \frac{4\pi G \delta \phi}{3} ; \delta_k = \frac{\delta \phi_k}{\delta}$

and local Friedmann equation : $2 + \delta H_k = \frac{8\pi G}{3} \delta \phi_k - \frac{1}{3} \frac{H^2 R_k}{a^2}$

From Poisson,

$$\delta \dot{\phi} = -\frac{H^2 \delta \phi}{4\pi G a^2} + \frac{2H \delta R_k}{4\pi G a^2}$$

Friedmann

$$\delta \dot{\phi} = \frac{2}{3H} \frac{4\pi G a^2}{\kappa^2}$$

Multiply by $\frac{2}{3H} \frac{4\pi G a^2}{\kappa^2}$,

$$\frac{2}{3} H^2 \delta \phi_k = \frac{2}{3} \frac{4\pi G a^2}{\kappa^2} \left( \frac{1}{2H} \frac{4\pi G a^2}{\kappa^2} \left( -\frac{2}{3} \frac{H^2}{a^2} (\delta_k + R_k) + \frac{2H \delta R_k}{4\pi G a^2} \frac{2}{3\pi G a^2} \right) \right)$$

$$\Rightarrow \frac{2}{3} H^2 \delta \phi_k = \delta_k \left( \frac{1}{3} - (1w) \frac{4\pi G a^2}{(\kappa^2)^2} \frac{2}{3} - 2 \right) + (1w) \frac{4\pi G a^2}{(\kappa^2)^2} \frac{2}{3} \frac{H^2}{a^2}$$

$$\Rightarrow \frac{2}{3} H^2 \delta \phi_k = \delta_k \left( \frac{1}{3} - (1w) \frac{4\pi G a^2}{(\kappa^2)^2} \frac{2}{3} - 2 \right) + (1w) \frac{4\pi G a^2}{(\kappa^2)^2} \frac{2}{3} \frac{H^2}{a^2}$$

$$\Rightarrow \frac{y}{3} - 2 - (1w) = \frac{y - 6 - 3w}{3} = \frac{5 + 3w}{3}$$
Using epoch when \( \omega = 0 \), the "growing mode" solution is to
(i.e. the equation above has an homogeneous
solution \( \Phi_k \sim a^{-5/3+w} \) which we ignore,
only take the particular
solution obtained from \( \Phi < 0 \))

Therefore \( \Phi_k = \frac{-3}{5+3w} (1+w) R_k \)
which is independent of time
of scale and order unity -
Equation (1) turns out to be valid
in during inflation, when \( \omega + 1 = \frac{\dot{a}}{a} \), \( \Phi \sim \Phi_k = \frac{-3}{2} (1+w) R_k \).

Therefore as a result, when perturbations cross both in the
bubble radius, they will do so as a fluctuation in the
varifational potential (The constant of proportionality depends
on whether the modes come back in Fast or MAT era).

Now we are ready to see how the power spectrum evolves some
inflation until after decoupling.

The linear power spectrum: Transfer Function

Unfortunately, Plot is not to
scale, RAD epoch should
span more in log(a)

Basic idea: for modes that
enter during MAT, we see
primordial perturbation, i.e.
\( \Phi_k \propto R_k \) when it crosses and
after that \( \Phi_k \) = const.

Note that mode in RAD are suppressed
So, from discussion above, we have that $\Phi_k = -\frac{3}{5+3w} R_k$

Hubble radius crossing. Remember that the power spectrum of the curvature perturbation is Harrison-Zeldovich (approximately, if $\eta_S = 1$):

$$P(\kappa) \propto k^{-3}$$

$P(\kappa)$ is for scales that are crossing now.

If scale that are a bit smaller, but still cross during the matter era, something happens since the gravitational potential stays constant for $\ln a = 1$ as we assume now) in linear perturbation theory,

$$-k^2 \Phi_k = \frac{3}{2} H^2 \delta_m \delta_k = \frac{3}{2} H^2 \delta \propto H^2 a \propto \frac{H_0^2}{a^2} \propto \frac{1}{a^2} \left( \frac{t_0}{a} \right)^3$$

Thus, for $k \leq k_{\text{eq}} = \left( \frac{H_0}{a_{\text{eq}} H} \right)^{-1} \mathrm{Mpc}^{-1}$, the comoving wavenumbers that cross Hubble radius at matter-radiation equality, we have that today,

$$P_{\Phi}(k) \propto P_R(k) \propto k^{-3} + (\eta_S - 1)$$

For modes that cross during RAD era, recall that density fluctuations grow only logarithmically (for modes that exist well before $z_R$). Then after they cross Hubble radius,

$$-k^2 \Phi_k = \frac{3}{2} H^2 \delta_k \propto H^2 a^2 \ln a \propto \frac{1}{a^2} \frac{a^2}{a^2} \ln a \propto \frac{a_{\text{eq}}}{a^2}$$

Therefore these perturbations are suppressed by a factor:

$$\ln \left[ \frac{a_{\text{eq}}}{a_{\text{eq}}(k)} \right] \left( \frac{a_{\text{eq}}(k)}{a_{\text{eq}}} \right)^2$$
By the time it reaches us now (after \( \Theta \), they stay constant \( n_0 \)), where \( a_0(k) \) is the scale factor at the time the mode \( k \) enters the Hubble radius.

The condition that enters is

\[
X_{\text{comoving}} = \frac{2 \pi}{k} \quad a_0(k) = H^{-1} [a_0(w)] \times \frac{1}{t} \times a_0^2(k) \]

\[
\Rightarrow \quad a_0(k) \propto \frac{1}{k} \quad \Rightarrow \quad \frac{a_0(k)}{a_0^0} = \frac{k_{\text{eq}}}{k}
\]

Then, the suppression factor is then,

\[
\ln \left( \frac{k_{\text{eq}}}{k} \right) \left( \frac{k_{\text{eq}}}{k} \right)^2 \quad (k_{\text{eq}} \leq k)
\]

So, for \( k > k_{\text{eq}} \):

\[
P_{\delta}(k) \propto P_0(k) \left[ \ln \left( \frac{k_{\text{eq}}}{k} \right) \left( \frac{k_{\text{eq}}}{k} \right)^2 \right]^2 \quad k > k_{\text{eq}}
\]

In addition, at scales small enough where free-streaming can take place and cause fluctuations, the power spectrum will be additionally suppressed,

\[
P_{\delta}(k) \rightarrow P_{\delta}(k) \mathcal{O} \left( \frac{\Theta}{k_{\text{eq}}} \right)^2
\]

where \( \Theta \) depends on the mass of the dark matter particle, recall that

\[
\Theta \approx 0.5 \, \text{Mpc} \quad (\approx 2 \times 10^9 \, \text{yr})^{1/3} \quad \left( \frac{m}{1 \, \text{keV}} \right)^{-4/3}
\]

We now put everything together. Since the density fluctuation is related to the gravitational potential through Poisson's equation, we have:
\[
\delta_k = -\frac{2}{3} \left( \frac{H}{\dot{H}} \right)^2 \delta_k = +\frac{2}{3} \left( \frac{H}{\dot{H}} \right)^2 \frac{3(1 + w)}{5 + 3w} R_k = \frac{2}{5} \left( \frac{H}{\dot{H}} \right)^2 R_k
\]

where, at large scales \( k \ll k_{eq} \), the power spectrum of density perturbations is,

\[
P_\delta(k) = \frac{4}{25} \left( \frac{H}{\dot{H}} \right)^4 P_\mathcal{R}(k) \propto k^4 R^{-3} (k^5 - 1) \propto k^n
\]

where \( n_5 \) is close to unity \( n_5 \) (Harrison-Zel'dovich) for inflationary perturbations. At smaller scales this behavior is changed by the suppression of growth during the epoch. It is customary to define a transfer function \( T(k) \), so that,

\[
P_\delta(k) = \frac{4}{25} \left( \frac{H}{\dot{H}} \right)^4 P_\mathcal{R}(k) T^2(k)
\]

where \( T(k) \) has the behavior:

\[
T(k) = \begin{cases} 
1 & k < k_{eq} \\
\ln \left( \frac{k_{eq}}{k} \right) & k_{eq} \lesssim k \lesssim k_{eq} \\
0 & k > k_{eq}
\end{cases}
\]

The power spectrum thus has the following shape (for \( H_2, n_5 = 1 \))

A couple of things to note,
At large scales we see the primordial power from inflation.

The characteristic scale \( \kappa_{eq} \) is the comoving Hubble radius at \( \tau = \tau_{eq} \) and depends on \( \Omega_m h^2 \) [if \( k \) is in units of \( h/\text{Mpc} \)], so this peak tells us about how much dark matter there is (when \( \Omega_m \gg 1/8 \)).

At scales \( k \gg k_{eq} \), where \( \Delta(k) \ll 1 \), non-linear connections to the power spectrum become important (remember we were using only linear PT so far).

At scales \( k \ll k_{eq} \), power is suppressed due to free streaming (though this can be regenerated by non-linear effects).

[Show current status from galaxy surveys]

I forgot (almost) to mention that a reasonable fit to the transfer function is given by the so-called BBKS form:

\[
T(q) = \frac{\text{ln}(1 + 2.34 q)}{2.34 q} \left[ 1 + 3.89 q + (16.1 q)^2 + (5.46 q)^3 + (6.71 q)^4 \right]^{-1/6}.
\]

where \( q = \frac{k}{\tau_{eq} h} \text{ Mpc}^{-1} \)

\[ T \equiv \Omega_m h \exp \left[ -\frac{\Omega_B}{\Omega_m} \left( \sqrt{\frac{2 \Omega_B}{\Omega_m}} - 1 \right) \right] \approx \Omega_m h \]

This is a numerical fit to a full numerical solution.

Note that it has the right asymptotics:

\[
T(q) \sim \begin{cases} 1 & q \to 0 \\ \frac{\text{ln}(q)}{q^2} & q \to \infty \end{cases}
\]