Tensor Perturbations

Now let's discuss the generation of perturbations during inflation in some detail, for a scalar field with some potential $V(\phi)$. We split into homogeneous part + perturbations,

$$\phi(x,t) = \phi_0(x,t) + \delta\phi(x,t)$$

From the energy-momentum tensor

$$T^\mu_{\nu} = \nabla^\mu \phi \nabla^\nu \phi - \left[ \frac{1}{2} \nabla^\mu \phi \nabla^\nu \phi - V(\phi) \right] \delta^\mu_{\nu}$$

And similarly decompose into background + linear in perturbations:

$$T^\mu_{\nu} = \langle T \rangle^\mu_{\nu} + \delta T^\mu_{\nu}$$

where

$$\langle T \rangle^0_0 = \frac{1}{2} \frac{\delta^2}{a^2} + V(\phi_0) = \rho_0$$

$$\langle T \rangle^0_i = 0$$

$$\langle T \rangle^i_j = \left[ - \frac{1}{2} \frac{\delta^2}{a^2} + V(\phi_0) \right] \delta^i_j = -\rho_0 \delta^i_j$$

where as you can see we have defined conformal time expansion by a dot.

For the perturbations we have

$$\delta T^0_0 = \frac{1}{a^2} \left[ - \frac{\delta^2}{a^2} A + \dot{\phi}_0 \delta \phi + V' a^2 \delta \phi \right] = \delta \rho$$

$$\delta T^0_i = \frac{1}{a^2} \dot{\phi}_0 \delta \phi ; i = (\rho + \dot{\phi}) (V_2 - \delta i)$$

$$\delta T^j_i = \frac{1}{a^2} \left[ \frac{\delta^2}{a^2} A - \dot{\phi} \delta \phi + V' a^2 \delta \phi \right] \delta_{j}^{i} = -\delta \rho \delta_{j}^{i}$$

A couple of things worth noting here. First from last equation we see that for a scalar field $\delta_{\phi} = 0$, so tensor modes will have no source. The second equation says that $\delta T^0_i$ is purely scalar mode,
so there is no source for vector modes, and since they obey a
wave equation, they will vanish (with appropriate ICs). Tensor
modes are the next easiest to figure out, so we will do this first
before we get into scalar modes (where we have to take into
account the coupling of $g$ and metric perturbations). Remember
from before we found that evolution of tensor modes obeys

$$(\partial^2 - k^2) \xi_{ij}^{\mathbf{T}} = 8\pi G a^2 \xi_{ij}^{\mathbf{T}} = 0$$

two source

for scalar field

Denote the two polarizations of $\xi_{ij}^{\mathbf{T}}$
by $h_+$ and $h_\times$, each polarization then evolves according to

$$\ddot{h} + 2\eta \dot{h} + k^2 h = 0$$

It is convenient to define:

$$\tilde{h} = \frac{M_{Pl}}{\sqrt{2}} \tilde{h}$$

The factor of $\sqrt{2}$ is to bring $h$ to a canonically normalized
scalar field, this is accomplished by writing the action for $h$ and
requiring its kinetic term have a factor $\sqrt{2}$. The $a^2$ factor is to get
rid of the $h^2$ term in the equation of motion, which faciliated the
mapping into a harmonic oscillator problem which is then trivial
to quantize. Then we have,

$$\ddot{h} + (k^2 - \frac{a^2}{2}) h = 0$$

The two solutions of this ODE are (assuming deSitter, i.e. $H = \text{const.}$ and $\dot{H} = 0$)

$$h = \frac{\mathcal{H}}{\sqrt{2} k} \left( \frac{k}{\mathcal{H}} + i \right) \mathcal{H}^{1/2} = \frac{1}{(2\pi)^{1/2}} \left[ 1 + \frac{\mathcal{H}}{2k} \right] e^{i k \mathcal{H}}$$

and $h_\times$. The solution with the right boundary condition is $h$, to
see this we require that in the small scale limit $\mathcal{H} \gg 1$ things
should reduce to the standard Minkowski result:

$$\xi \rightarrow 2^{3/2} - i F_{\nu} e^{-i F_{\nu} t}$$

or

$$\xi \rightarrow 2^{1/2} e^{-i F_{\nu} t}$$

$\xi_{ij}^{\mathbf{T}}$
where $a \sim V^{-1/2}$, with $V$ the volume of the box, and for a massless field (like the graviton) $E_{\mathbf{k}} = k/a$. Now, since

$$
\frac{d}{dt} \left( \frac{k}{a} \right) = -\frac{k}{H} \frac{a}{a^2} = -\frac{k}{a} \quad \text{(here $\dot{a}$ is $d/dt$)}
$$

Then $\frac{k}{a} \sim \frac{k}{a} \dot{a}$ so $V$ has the right asymptotics.

Now that we have the right modes we can quantize it, we have

$$\hat{\psi} = \sum_{\mathbf{k}} \hat{a}^\dagger \psi_{\mathbf{k}} + \psi_{\mathbf{k}} \hat{a}^\dagger$$

where $\hat{a}$ and $\hat{a}^\dagger$ are annihilation and creation operators for modes of wavevector \( \mathbf{k} \) with standard commutation relations $[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger] = \delta_{\mathbf{k}, \mathbf{k}'}$. Now if we assume we are in the vacuum state we have \( \hat{\psi} = 0 \) for the power spectrum of $\hat{\psi}$:

$$\langle 0 | \hat{\psi}^\dagger \mathbf{k} \hat{\psi} \mathbf{k}^\prime 10 \rangle = \left( \frac{2\pi}{k} \right)^2 \delta_{\mathbf{k}, \mathbf{k}^\prime} \delta (E_{\mathbf{k}} - E_{\mathbf{k}^\prime}) = \frac{P_{\mathbf{k}}}{k} \delta (E_{\mathbf{k}} - E_{\mathbf{k}^\prime})$$

The assumption of a vacuum state is a reasonable one, since if one had significant energy in the fluctuations (e.g. due to large occupation numbers) one would not have inflation win the first place, one needs to be dominated by the smooth potential energy, gradient terms violate the condition $\langle \phi^2 \rangle \sim f_\mathbf{k}$. If inflation starts long before the modes we access today, where crossing Hubble during inflation (the bare minimum to solve horizon problem) then these modes are high-frequency enough that any mean-zero occupation number would be a strong destruction contribution to the perturbations, thus it is natural to choose a vacuum state.

Note however, that we have made the choice of the vacuum for a richer space, in practice in that is not exactly the siltor (1)}
has to end at some time, so it is not exactly constant), so there may be connections to the mode functions, which are expected to be small, but it is hard to say more quantitative statements.

Mapping back to $k$ we have for the power spectrum of tensor modes:

$$P_h(k) = \frac{16 \pi G}{a^2} |h|^2 = \frac{16 \pi G}{a^2} \frac{H_k}{2k^3} \left(1 + \frac{k^2}{H_k^2}\right)$$

Note again that at small scales, $k \gg H$, then $P_h \sim \frac{1}{k a^2}$, the Minkowski result (with adiabatic adjustment to scale factor), for $k \gg H$ at the scales cross the Hubble radius, $P_h(k)$ becomes

$$P_h(k) = \frac{16 \pi G}{k^3}$$

This is for each polarization, so on top of this we should multiply by a factor of 2, thus the amplitude of the gravitational wave spectrum is

$$P_{GW}(k) = 2 \frac{H_k^2}{M_{pl}^2} k^3$$

$$M_{pl}^2 = \frac{1}{8\pi G}$$

Notice that the amplitude is proportional to $H_k^2$, which is proportional to the potential energy $V$, thus a measure of this amplitude will give a measure of the energy scale during inflation, which would be very interesting!

Scalar Perturbations:

The case of scalar perturbations is more complicated because one in principle needs to deal with the scalar perturbations of the metric and the inflaton field simultaneously. However, simplification is possible by choosing a gauge in which they decouple. That happens if we choose the slicing so that the spatial part of the
metric has no perturbations, this is known as spatially flat gauge. Here we have for the scalar part of the metric

$$ds^2 = -(1 + 2\Phi) \, dt^2 - 2a \sum_i dx_i \, d\Phi + \delta_{ij} \, dx_i \, dx_j$$

In this gauge it follows that to leading order in the slow-roll parameters the evolution of $\delta \Phi$ is the same as a free-scalar field, (show it!)

$$\ddot{\delta \Phi} + 2H \dot{\delta \Phi} + k^2 \delta \Phi = 0$$

the same as for a GW mode. Then, its power spectrum is

$$P_{\delta \Phi} = \frac{H^2}{2k^3}$$

Now, we would like to map this result into a result of the power spectrum of scalar metric fluctuations, since that's what we measure (i.e. today $\phi$ is gone). To go from this to conformal Newtonian gauge we need to do a gauge transformation, and a trick. The idea is to identify a gauge-invariant variable that reduces to $\delta \Phi$ when we are in the spatially-flat gauge, and then do a gauge transformation to the conformal Newtonian gauge to see what that implies for the gravitational potential (or more properly) curvature fluctuation spectrum.

From the gauge transformation of scalar part of metric, you can check that the following "Bardeen potential" is a gauge invariant,

$$\Phi_A = \frac{A}{a} - \frac{1}{a} \frac{2}{3} [a(\dot{E} + B)]$$

$$\Phi_H = \frac{\sqrt{E} - D}{3} + \mathcal{H} (B + \dot{E})$$

And, similarly, for the stress-energy variable

$$\nabla = \frac{\dot{E} + B}{a} \frac{k^2}{k^2 \nu^2}$$
\[ \xi_m = -1 - \frac{T_0}{g} + \frac{3H}{k^2 g} k^2 T^0 \]

Now, in the spatially flat gauge and for our parameters we have

\[ \begin{align*}
\nu &= \frac{i k \phi_0 \delta \phi}{\Theta (k^2 + \phi^2) a^2} \\
\Phi_H &= \nabla \Phi
\end{align*} \] 

(spatially flat gauge)

Since \( D = E = 0 \) in this gauge, a linear combination of gauge invariant variables is gauge invariant, in particular the following combination

\[ \xi = -\Phi_H - \frac{i a H \nu}{k} \]

gives us something proportional to \( \delta \dot{\Phi} \) in the spatially flat gauge,

\[ \xi = -\frac{a H}{\phi_0} \frac{\dot{\phi}_0}{(a^2 (k^2 + \phi^2))} = -\frac{a H}{\phi_0} \delta \dot{\Phi} \] 

(spatially flat gauge)

Therefore, we can immediately write down the power spectrum of \( \xi \):

\[ P_\xi = \left( \frac{a H}{\phi_0} \right)^2 \frac{P_{\delta \Phi}}{a^3} = \left( \frac{2 \pi^2 G}{k^3} \right) \frac{1}{8} \]

\[ (V')^2 = \frac{9H^2 (d\phi/dt)^2}{5M_{\text{Pl}}^4 H^4} = \frac{\dot{\phi}_0^2}{a^2 H^2} \frac{1}{H^2} = \frac{2 k^2}{H^2} \]

The interesting piece of information is that \( \xi \) is related to the Newtonian potential, the scalar potential of the gravitation potential, by (remember from Post-32)

\[ \phi = \begin{cases} 
\xi - \frac{3 \pi}{2} & \text{real} \\
\frac{-3 \pi}{2} & \text{imaginary (in conformal)}
\end{cases} \]

So we can interpret the power spectrum of \( \xi \) as the power spectrum of the gravitational potential. In the comoving gauge, where velocities are zero, \( \xi = -\frac{\dot{\Phi}}{H} \) and since the curvature due to fluctuation is \( \frac{4k^2 \delta \dot{\Phi}}{a^2} \), in such a gauge \( \xi \) can be interpreted as curvative fluctuations, that's why we say that inflation generates...
Therefore, after inflation, and during the slow-roll era we have for the potential $\Phi$ power spectrum,

$$P_{\Phi} = \frac{8\pi G H^2}{9 k^3}$$

where we remember the Hubble constant is evaluated at the time the scale $k$ crosses the Hubble radius $H a = k$.

The results for power spectra can be summarized in terms of dimensionless amplitudes

$$\Delta(k) \equiv \frac{4\pi^2 k^3 P(k)}{2\pi^3} \longrightarrow \text{this is to account for } \frac{1}{(2\pi)^3} \text{in } P(k),$$

$$\Rightarrow \Delta_{GW}(k) = \frac{H^2}{16\pi^2 M_{pl}^2} \frac{1}{\epsilon} \left( \text{or } \frac{H^2}{16\pi^2 M_{pl}^2} \frac{1}{\epsilon} \right)$$

We see that to lowest order these do not depend on $k$, only through $H$ (or $V$) they depend weakly, since $H$ (and $V$) are slowly varying during inflation. We define the spectral indices for scalar and tensor fluctuations as

$$n_s(k) - 1 = \frac{d \ln \Delta_{\Phi}}{d \ln k} = \frac{d \ln V}{d \ln k} - \frac{d \ln \epsilon}{d \ln k} \quad \text{(or) } 2 \frac{d \ln H}{d \ln k} - \frac{d \ln \epsilon}{d \ln k}$$

$$n_t(k) = \frac{d \ln \Delta_{GW}}{d \ln k} = \frac{d \ln V}{d \ln k}$$

We transform derivative $d \ln k$ to $d \phi$ using that close to Hubble crossing,

$$d k = \frac{H d a}{f} \Rightarrow \frac{H a d t}{f} = \frac{H^2 a}{f} d \phi = \frac{H}{\sqrt{-V}} \frac{d \phi}{d H} \Rightarrow 3H \frac{d \phi}{d H} = -\sqrt{-V}$$

$$\Rightarrow \frac{d \ln k}{d \phi} = -\left( \frac{AH}{f} \right) \frac{3H^2}{\sqrt{-V}} d \phi = -\frac{1}{\sqrt{-V}} \frac{1}{M_{pl}^2} d \phi$$
\[
\frac{a}{L} = - M \eta^2 \frac{V}{V'} \frac{d}{d\phi}
\]

Recall: \( \xi = \frac{M \eta^2 (V')^2}{2} \)

\( \eta \equiv \frac{\eta^2}{V} \)

\[
\frac{d \ln V}{d \ln k} = - \frac{M \eta^2}{V} \frac{d \ln V}{d \phi} = - 2 \xi
\]

and
\[
\frac{d \ln \xi}{d \ln k} = - \frac{M \eta^2}{V} \frac{d \ln \xi}{d \phi} = - M \eta^2 \frac{V'}{V} \frac{V'' - V'''}{V} = - 2(\eta - 2 \xi) = 4 \xi - 2 \eta
\]

Then, we have:
\[
\begin{align*}
\eta_0 (k) - 1 &= 2 \eta - 6 \xi \\
\eta_T &= - 2 \xi
\end{align*}
\]

The reason for the asymmetry in the definitions of \( \eta_0 \) and \( \eta_T \) is as follows. Both \( \eta_0 - 1 \) and \( \eta_T \) measure deviation from "scale invariance", but this we mean independent of scale \( k^2 P(k) \)'s for scalar and tensor modes. Thus, \( \eta_0 - 1 \) and \( \eta_T \) are called scalar and tensor "tilt" of the spectrum. The reason for using \( \eta_0 - 1 \) as definition rather than \( \eta_T \) is that \( \eta_T \) defined as above becomes the spectral index (at large scales) for density perturbations.

Due to Poisson equation, \( k^2 \Phi \sim \delta \Rightarrow k^4 P_{\Phi} \sim P_\delta \)

\[
\Rightarrow P_{\Phi} \sim \frac{k^{2 \eta - 1}}{k^3} \sim k^{\eta}
\]

Note that if one would detect gravitational waves, the ratio of tensor to scalars is proportional to \( \xi \), which in turn is proportional to the tensor tilt \( \eta_T \). These two relationships can be written (in terms of potential \( \Delta_\Phi = \frac{\partial \Phi}{\partial \phi} \))

\[
\left\{ \begin{array}{l}
\frac{\Delta \Phi}{\Delta \phi} = 8 \xi = - 4 \eta_T \\
\end{array} \right.
\]

Note: We instead of "common" 46 appears here because we included the 2 polarizations into \( \Delta \Phi \). This is the so-called consistency relation, if we get to observed it, it would be a strong argument in favor of inflation.