Weak lensing by Galaxy Clusters

If the density of a cluster is below \( \Sigma_0 \), or if we study regions of it outside the giant arcs where \( \kappa < 1 \), so that there are no multiple images, we are pushing the weak lensing regime. In this case, although difficult to detect directly the convergence \( \kappa \), the effects of shear \( \gamma \) can become observable: image as magnified by

\[
\lambda_1 = (1 - \kappa - \gamma)^{-1}
\]

in one direction and

\[
\lambda_2 = (1 - \kappa + \gamma)^{-1}
\]

in the other orthogonal direction, so shear induces tangential deformation of the images—pictorially,

Thus, a circular image will be distorted into an ellipse of ratio

\[
r = \frac{\lambda_2}{\lambda_1}
\]

and thus the average ellipticity induced would be

\[
\langle \epsilon \rangle = \frac{1 - r}{1 + r} = \frac{\gamma}{1 - \kappa}
\]

Thus when the convergence is small, \( \kappa \ll 1 \), the average ellipticity gives a direct measure of the shear—however, since the shear (as well as convergence) are given by second derivatives of the projected potential, one can perform inversion procedures to recover \( \kappa \) from \( \gamma \) and thus the surface density profile \( \Sigma = \kappa \Sigma_0 \) can be reconstructed.

But galaxies are not circles, so what can one learn by measuring ellipticities of galaxies which already at the unperturbed by lensing level are not round? The point is that by studying lots and lots of background galaxies one should on average notice a pattern of tangential shear, so while each galaxy by itself is elliptic the orientation of this ellipse will be random and will circularize upon average, while the tangential shear will not—That's the
To show how to go from $\gamma$ to $\alpha$, we write a complex shear (related to complex elliptic $\frac{1}{2} \varepsilon = \varepsilon_1 + i \varepsilon_2$)

$$T = \alpha_1 + i \alpha_2 = \left[ \frac{1}{2} \left( \frac{1}{\sigma_1^2} - \frac{1}{\sigma_2^2} \right) + i \frac{2}{\sigma_1 \sigma_2} \right] \Psi$$

where $\Psi = \frac{2}{c^2} \frac{D L_0}{D \sigma_0 D \sigma_0}$ is a scaled version of the projected potential.

Since $\int_{\Omega_1} \frac{1}{2} \omega_1 \omega_2 \mathcal{E}(\Omega_1) \ln |\Omega_1 - \Omega_1'|$, \( \mathcal{F}(\Omega) = \frac{4 G D L_0}{c^2 D \sigma_0 D \sigma_0} \int d^2 \psi \mathcal{E}(\psi) \ln |\psi - \psi'|$

$$\Rightarrow \Psi(\Omega) = \frac{1}{\pi} \int d^2 \psi' \mathcal{E}(\psi') \ln |\psi - \psi'|$$

$$\Rightarrow T = \frac{1}{\pi} \left[ \frac{1}{2} \left( \frac{1}{\sigma_1^2} - \frac{1}{\sigma_2^2} \right) + i \frac{2}{\sigma_1 \sigma_2} \right] \int d^2 \psi' \mathcal{E}(\psi') \ln |\psi - \psi'|$$

Let $D(\psi) = \left[ \frac{1}{2} \left( \frac{1}{\sigma_1^2} - \frac{1}{\sigma_2^2} \right) + i \frac{2}{\sigma_1 \sigma_2} \right] \ln |\psi| = \frac{-4}{(\sigma_1 - i \sigma_2)^2}$

$$\Rightarrow T = \frac{1}{\pi} \int d^2 \psi' \mathcal{E}(\psi') D(\psi - \psi')$$

This convolution by kernel $D$ can be inverted by going to Fourier space:

$$\mathcal{F} \left[ D \right](\Omega) = \int d^2 \psi \mathcal{E}(\psi) e^{i \Omega \cdot \psi} = \pi \frac{\kappa_1^2 - \kappa_2^2 + 2i \kappa_1 \kappa_2}{\lambda^2}$$

with $D(\Omega) \mathcal{D}^*(\Omega) = \pi^2 - $ By convolution theorem, $T(\Omega) = \frac{\mathcal{F}(\Omega) \mathcal{D}(\Omega)}{\pi}$

$$\Rightarrow \mathcal{F}(\Omega) = \frac{D^*(\Omega) T(\Omega)}{\pi}$$

Note that a constant convergence appears here, since a \textit{thin slice} of uniform density cannot be determined from measuring the shear alone. Note that the deconvolution to get $\mathcal{F}$ from $T$ is not local (involving integration over all angle) so boundary conditions must be properly taken care of.
when $x$ is not negligible, then $T = (1-x) \langle E \rangle$ and this decoupling leads to an integral equation (now $x$ is also involved inside integral) which can be used to solved for $x$ by iteration. To break the "mass sheet degeneracy" and determine $\Delta \sigma$ it is necessary to measure the effects of the cluster convergence, which can be done if the magnification of the images can be also determined.

**Cosmic shear**

Cosmic shear refers to the effects of weak lensing on random parts of the universe, i.e. not looking specifically at lenses such as cluster of galaxies. In this case the lensing is caused by the large-scale inhomogeneities in the distribution of perturbations in the Universe. Although the effect is rather small (induced galaxy ellipticities of a few percent), averaging over $10^5$ of galaxies on few sq.deg can be used to extract the signal. The important thing is that such measurement traces the spectrum of dark matter perturbations, so its study can constrain the power spectrum of matter in the universe and other cosmological parameters.

To calculate cosmic shear one has to go beyond the thin lens approximation, since the bending of the light ray takes place all along the photon's path. To do so, one has to integrate the geodesic equation in a perturbed universe, and the angular deflection will depend on the transverse gradient of the gravitational potential integrated along the line of sight. Let's sketch how this is done, and what can we learn from it.

The effect of density perturbations is described by the geodesic equation for a perturbed Robertson-Walker metric, which we work in the conformal Newtonian gauge,
\[ ds^2 = a^2 \left[ (1 + \frac{\tau}{c^2}) d\tau^2 - (1 - 2\frac{\tau}{c^2}) \left( d\chi^2 + r^2 \sin^2 \theta \, d\phi^2 \right) \right] \]

where \( r(\tau) \) is the comoving angular distance given last class. The metric perturbations are described by the Newtonian potential \( \phi \) and the curvature potential \( \Psi \) (These are \( A = \phi \), \( D = -\Psi \) in notation used earlier in the course).

We are doing this calculation to first order in \( \phi, \Psi \). We take a freewill direction (say \( y \)) along the light beam and take angular coordinates \( \theta \) with \( \alpha = 1, 2 \) corresponding to \( x, y \), i.e., \( d\theta_1^2 + d\theta_2^2 = d\theta_1^2 + d\theta_2^2 = d\theta^2 + \sin^2 \theta \, d\phi^2 \). Taking the affine parameter to be the comoving separation \( i.e., \, d\alpha = dx \) (since we are working to first order), we have the geodesic equation

\[
\frac{d^2 \theta_a}{dx^2} = -\frac{2}{r} \frac{d}{dx} \frac{d\theta_a}{dx} - \frac{2\Psi_a}{r^2}
\]

where we defined the effective potential \( \Psi \equiv \frac{1}{2} (\phi + \Psi) \) and \( \Psi_a \equiv \frac{d\Psi}{d\theta_a} \).

Note that in GR, for dark matter where there is no anisotropy stress \( \Psi = \nu \), the Newtonian potential \( \Psi \) is the Newtonian potential. In theories where gravity is modified to explain cosmic acceleration \( \phi + \Psi \), and thus cosmic shear will be sensitive to this. After some algebra, this equation can be integrated to read

\[
\Theta_a(\chi) = \Theta_a^{(0)} - \frac{2}{c^2} \int_{\chi}^{\chi'} d\chi' \, \Psi_a \left[ \bar{r} \, r(\chi'), \chi' \right] \frac{r(\chi, \chi')}{r(\chi) \, r(\chi')}
\]

Notice that the integral involves the perturbed path itself \( \bar{r} \). To first post-Newtonian order, we can write \( \bar{\Theta}(\chi) = \bar{\Theta}^{(0)} \, r(\chi) \) and then we can write the amplification matrix as

\[
A^{-1}_{ij} = S_{ij} - \frac{2}{c^2} \int_{\chi_0}^{\chi} d\chi' \, \Psi_{ij} \left[ \bar{r} \, r(\chi), \chi' \right] \frac{r(\chi, \chi')}{r(\chi)}
\]

so the role of the scaled version of the projected potential is played by

\[
\bar{\Psi} = \frac{2}{c^2} \, \frac{1}{r(\chi)} \int_{\chi_0}^{\chi} d\chi' \, \Psi \left[ \bar{r} \, r(\chi), \chi' \right] \frac{r(\chi, \chi')}{r(\chi')}
\]
The convergence is given then by
\[ \chi(q) = \frac{1}{2} \sq{ \varphi} \sq{ \varphi} = \frac{1}{c^2} \int_0^\chi dq' \int \delta \left( \sq{ r} - \sq{ q} \right) \frac{r(q-q')r(q')}{r(q)} \delta (\sq{ r} - \sq{ q}) \]

The derivatives are with respect to the coordinates \( \sq{ r} = \varphi_1 q_1 \) - We can write the transverse Laplacian \( \nabla^2 = \nabla^2 - \nabla^2 \partial_\perp \), and it is possible to show upon integration that the radial Laplacian does not contribute to \( \chi \). Then, using the Poisson equation,
\[ \nabla^2 \varphi = \frac{3}{2} \frac{H_0^2 \sin^2 \frac{\chi}{a(\chi)}}{a(\chi)} \delta (\sq{ r} - \sq{ q}) \]

Note here we have assumed \( \Omega_R \), otherwise one has to relate \( \nabla^2 (\varphi + \chi) \) to \( \delta \) using the appropriate modification of Poisson equation. The result will still be some relationship between the convergence \( \chi \) and the line of sight projection of \( \Omega_R \)

Then we can write:
\[ \chi = \frac{3}{2} \frac{\sin^2 \frac{\chi}{a(\chi)}}{a(\chi)^2} \int_0^\chi dq' \frac{\varphi(q-q') \varphi(q')}{\varphi(q)} \delta (\sq{ r} - \sq{ q}) \]

Defining \( w(q) = \frac{3}{2} \frac{\sin^2 \frac{\chi}{a(\chi)}}{a(\chi)^2} \frac{\varphi(q-q') \varphi(q')}{\varphi(q)} \frac{1}{a(q')} \)

We can simply write:
\[ \chi = \int_0^\chi w(q) \delta (\sq{ r} - \sq{ q}) dq' \]

where \( w(q) \) is a \( 2D \) weight along the line of sight - Since distances are proportional to \( a(q) \), note that \( \chi \) does not depend explicitly on \( a(q) \). But it is proportional to \( \Omega^* \), and has cosmological information through the ratio of comoving distances, which effectively measures the lensing efficiency along the line of sight.

Let's now consider the power spectrum of lensing fields. We define the 2D transform,
\[ \xi(\mathbf{x}) = \int \frac{d^2 p}{(2\pi)^2} \bar{\xi}(\mathbf{p}) e^{-i\mathbf{p}\cdot \mathbf{x}} \]  

\[ \langle \xi(\mathbf{x}) \xi(\mathbf{x}') \rangle = P_x(\mathbf{x}) P_x(\mathbf{x}') \]  

We assume one observes a small patch of the sky so that we can treat flats by approx. and Fourier expand instead of \( \xi(\mathbf{x}) \).

\[ \langle \xi(\mathbf{x}) \xi(\mathbf{x}') \rangle = \int \frac{d^2 k}{(2\pi)^2} P_x(k) e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}') \langle \delta(\mathbf{p}_0, \mathbf{x}) \delta(\mathbf{p}_0, \mathbf{x}') \rangle \]  

\[ = \int d\mathbf{x}^1 d\mathbf{x}^2 w(\mathbf{x}^1) w(\mathbf{x}^2) \int d^3 \mathbf{k} P_8(\mathbf{k}) e^{i \mathbf{k} \cdot (\mathbf{x}^1 - \mathbf{x}^2)} \]  

The power spectrum, related to the two-point correlation function through

\[ \langle \xi(\mathbf{x}) \xi(\mathbf{x}') \rangle \]  

Let's now compare the \( k_x \) and \( k_t \) values that dominate the integral in the small-angle approximation (i.e. \( \| \mathbf{k} \| \ll \text{small} \)).

The integral over \( k_x \) is dominated by \( k_x \ll k_t \) where \( d\mathbf{x} \) is the typical scale of variation of \( w(\mathbf{x}) \). On the other hand, for \( k_t \) we have instead that \( k_t \ll k_x \ll k_x \) where we used \( r \). Then in small angle approx, we have \( k_t \ll k_x \). Then integral over \( k_x \) sets \( x = x^1 \) and we obtain

\[ \langle \xi(\mathbf{x}) \xi(\mathbf{x}') \rangle \propto 2\pi \int d\mathbf{x} w^2(\mathbf{x}) \int d^2 k_x P_8(k_x) e^{i \mathbf{k}_x \cdot (\mathbf{x} - \mathbf{x}')} \]  

\[ = 2\pi \int d\mathbf{x} w^2(\mathbf{x}) \int d^2 \mathbf{k} P_8(\frac{\mathbf{k}}{r_\text{med}}) e^{i \mathbf{k} \cdot (\mathbf{x} - \mathbf{x}') \langle \delta(\mathbf{p}_0, \mathbf{x}) \delta(\mathbf{p}_0, \mathbf{x}') \rangle} \]  

which means that the convergence power spectrum can be written as

\[ P_x(k) = 2\pi \int d\mathbf{x} \frac{w^2(\mathbf{x})}{r^2(\mathbf{x})} P_8 \left( \frac{\mathbf{k}}{r_\text{med}} \right) \]  

\[ \text{which can be evaluated.} \]
Scalar and Pseudo-scalar shear: E & B modes

Let's go back to the definition of convergence and shear and the amplification matrix - Recall that

\[
A_{ij} = \begin{bmatrix}
1 - \alpha - \gamma_1 & -\delta_2 \\
-\delta_2 & 1 - \alpha + \gamma_1
\end{bmatrix} \Rightarrow \text{in the weak lensing approximation the amplification matrix reads,}
\]

\[A_{ij} = \begin{bmatrix}
1 + \alpha + \gamma_2 & \gamma_2 \\
\gamma_2 & 1 + \alpha - \gamma_1
\end{bmatrix} = \delta_{ij} + \begin{bmatrix}
\alpha & 0 \\
0 & \alpha
\end{bmatrix} + \begin{bmatrix}
\gamma_1 & \gamma_2 \\
\gamma_2 & -\gamma_1
\end{bmatrix}
\]

Now, since \( \alpha \) and \( \gamma \) are derived from the same underlying projected potential, they are related to each other, as we saw:

\[
T^\prime(\xi) = \frac{\alpha(\xi) \Delta(\xi)}{l^2} = \frac{\delta_1^2 - \delta_2^2 + 2\delta_1 \delta_2}{l^2} \chi(\xi)
\]

\[
\delta_1 + i\delta_2 \Rightarrow \delta_1^2 = \frac{\delta_1^2 - \delta_2^2}{\chi^2} \chi \quad \delta_2^2 = \frac{2\delta_1 \delta_2}{\chi^2} \chi \Rightarrow \begin{bmatrix}
\delta_1^2 - \delta_2^2 \\
2\delta_1 \delta_2
\end{bmatrix} = \begin{bmatrix}
(\delta_1^2 - \delta_2^2) \chi + 2\delta_1 \delta_2 \chi \\
2\delta_1 \delta_2 \chi
\end{bmatrix} = \frac{(\delta_1 - \delta_2)^2 \chi + 4\delta_1 \delta_2 \chi}{\chi^2}
\]

Then this can be written in real space as

\[
\nabla^2 \chi = (\delta_1^2 - \delta_2^2) \chi + 2\delta_1 \delta_2 \chi
\]

If we look at the shear matrix, we see that it has two degrees of freedom which are the two polarization states of a gravitational wave, in fact, the shear can be thought of as a spin-2 field in the sky - the \( \delta_1 \) corresponds to the + polarization, and \( \delta_2 \) to the \( \times \) polarization, just rotated by 45° with respect to the + polarization. We can write this matrix also as

\[
\gamma_{ab} = \begin{bmatrix}
\gamma_1 & \gamma_2 \\
\gamma_2 & -\gamma_1
\end{bmatrix} = \delta \begin{bmatrix}
\cos 2\alpha & \sin 2\alpha \\
\sin 2\alpha & -\cos 2\alpha
\end{bmatrix}
\]

Where \( \gamma = \gamma_1^2 + \gamma_2^2 \) is the amplitude of the shear and \( \alpha \) is the direction of the shear with respect to some fiducial direction in the sky. Notice that directionality is modulo \( \pi \), not \( 2\pi \), since stretching an image in one direction is exactly the
opposite as stretching in the opposite direction. Thus, if one wants to represent shear graphically, one may use "rods", or "arrow-less" vectors, to emphasize this modulo-$\pi$ property of spin-2 fields. For the same reason, rotating by $90^\circ$ changes the sign of the shear, as stretching in one direction is equivalent to compressing in the perpendicular direction.

The shear tensor field can be decomposed into a scalar and pseudo-scalar mode (reminiscent of scalar and vector modes, or gradient & curl modes in 3D). We write

$$\nabla^2 E = (\partial_a \partial_b - \frac{\nabla^2}{2} \delta_{ab}) \chi_{ab} \quad \text{(Scalar)} \quad \text{"E-mode"}$$

$$\nabla^2 B = \frac{1}{2} \left( \partial_a \delta_c \chi_{cb} + \partial_b \delta_c \chi_{ca} \right) \chi_{ab} \quad \text{(pseudo-scalar)} \quad \text{"B-mode"}$$

where $\chi_{ab} = (0 \quad 1) \quad \text{is the Levi-Civita symbol in 2D}$. These can be written more explicitly as

$$\begin{cases}
\nabla^2 E = (\delta_i^2 - \delta_x^2) \chi_i + 2 \delta_i \partial_2 \delta_{i2} \\
\nabla^2 B = -2 \delta_1 \partial_2 \chi_2 + (\delta_1^2 - \delta_x^2) \delta_{12}
\end{cases}$$

Notice that the E-mode is no other than the convergence $\chi$!

What about the B-mode? Converting to Fourier space we see that

$$\mathcal{B} = -2\ell_1 \ell_2 \frac{\ell_1^2 - \ell_2^2}{\ell_1^{\prime}} \chi + (\ell_1^2 - \ell_2^2) \frac{2\ell_1 \ell_2}{\ell_1^2} \chi = 0 \quad \text{!}$$

So lensing by density perturbations produces no B-mode! \[\text{This can be generally} \quad \text{for vector/tensor perturbations}\]

This constitutes an important symmetry property of lensing.

That means one can write $\chi_{ab}$ in terms of E-mode only, indeed

$$\chi_{ab} = 2 \frac{\delta_a \partial_b - \delta_{ab}}{\ell_1^2} \chi$$

which is another way of writing $\chi_{ab}$, completely equivalent to what we had before.
What do the shear patterns look like?

For E-mode one can represent it as (about a point)

\[ \text{pattern} \quad \text{or} \quad \text{pattern} \quad (E\text{-mode}) \]

Note the second pattern is the first rotated by 90° (opposite sign). For B-mode,

\[ \text{pattern} \quad \text{or} \quad \text{pattern} \quad (B\text{-mode}) \]

i.e. the same pattern as E but rotated by 45° (with vector arrows)

Note the parity of these modes is opposite - i.e., inverting about the origin, the E-modes are invariant (positive parity), while the B-modes change by a sign, i.e. converts between the left and right patterns (negative parity).

For these reasons correlators of E & B are very simple:

\[ \langle EE \rangle \quad \Rightarrow \quad \text{powerspectrum \& E = } \mathcal{E} \quad \text{has information on } P_\mathcal{E} \]

\[ \langle EB \rangle = 0 \quad \text{by parity symmetry (cross-correlation of positive wavelength)} \]

\[ \langle BB \rangle \quad \text{is parity even, but it vanishes for lensing by scalar modes (can be generated by vector/tensors but usually very small)} \]