1. Consider a change of coordinates $x^\mu \rightarrow x^\mu + \xi^\mu$.

a) Using the invariance of the interval show that the metric undergoes the transformation

$$g_{\mu\nu} \rightarrow g_{\mu\nu} - \xi^\alpha_{,\mu} g_{\alpha\nu} - \xi^\alpha_{,\nu} g_{\mu\alpha} - g_{\mu\nu,\alpha} \xi^\alpha,$$

(1)
to first order in metric perturbations and $\xi$.

b) Use the SVT decomposition of the gauge transformation and the metric perturbations to show that

$$A \rightarrow A - \dot{\alpha} - \mathcal{H} \alpha, \quad B \rightarrow B - \alpha + \dot{\beta}, \quad D \rightarrow D - \frac{1}{3} \nabla^2 \beta - \mathcal{H} \alpha, \quad E \rightarrow E - \beta,$$

(2)

$$B_i^V \rightarrow B_i^V - \dot{\epsilon}_i, \quad E_i^V \rightarrow E_i^V - \epsilon_i, \quad E_{ij}^T \rightarrow E_{ij}^T$$

(3)

where dots are derivatives with respect to conformal time and the SVT decomposition of the gauge transformation reads $\xi^0 \equiv \alpha$ and $\xi^i = \nabla^i \beta + \epsilon^i$ with $\nabla_i \epsilon^i = 0$ (transverse), while for the metric

$$ds^2 = a^2 \left\{ - (1 + 2A)d\tau^2 - 2B_i d\tau dx^i + \left[ (1 + 2D)\delta_{ij} + 2E_{ij} \right] dx^i dx^j \right\}$$

(4)

where

$$B_i = \nabla_i B + B_i^V, \quad E_{ij} = D_{ij} E + \frac{1}{2} (\nabla_i E_j^V + \nabla_j E_i^V) + E_{ij}^T,$$

(5)

with $D_{ij} = \nabla_i \nabla_j - \frac{1}{3} \delta_{ij} \nabla^2$, $B_i^V$ and $E_i^V$ transverse, and $E_{ij}^T$ transverse and traceless.

c) Consider only scalar perturbations. Show that from an arbitrary gauge one can always go to the conformal Newtonian gauge ($B = E = 0$) or to the spatially-flat gauge ($D = E = 0$) and in both cases there is no additional coordinate ambiguity.

2. Show using stress-energy conservation that in the spatially-flat gauge, the equations of motion for the inflaton perturbations correspond to a free field

$$\ddot{\delta \phi} + 2\mathcal{H} \dot{\delta \phi} + k^2 \delta \phi = 0,$$

(6)
to leading order in the slow-roll parameters.