Homework Set #2 (Due 4/8)

1. *Derivation of the suppression of the scale-invariant power spectrum due to evolution in the radiation era.* Assume only dark matter and radiation are present, and work at scales below Hubble radius \((k \gg \mathcal{H})\), where radiation perturbations can be neglected \(\delta_R \approx 0\).

   a) Starting from conservation of stress energy, show that the equation of motion for dark matter density perturbations can be written in linear perturbation theory as,

   \[
   \frac{\partial^2 \delta}{\partial \tau^2} + \mathcal{H} \frac{\partial \delta}{\partial \tau} \approx 4\pi G a^2 \bar{\rho}_M \delta, \tag{1}
   \]

   where \(\tau\) is conformal time, \(\mathcal{H} = \frac{d \ln a}{d \tau}\), and \(\bar{\rho}_M\) is the average matter density.

   b) Using the Friedmann equation \(\mathcal{H}^2 = \frac{8\pi G a^2 (\bar{\rho}_M + \bar{\rho}_R)}{3}\), and defining a new time variable, \(x \equiv \frac{a}{a_{EQ}}\), where \(a_{EQ}\) denotes the scale factor at matter-radiation equality, i.e. \(\bar{\rho}_M(a_{EQ}) = \bar{\rho}_R(a_{EQ})\), show that Eq. (1) can be rewritten as,

   \[
   2x(1 + x) \frac{\partial^2 \delta}{\partial x^2} + (3x + 2) \frac{\partial \delta}{\partial x} = 3 \delta \tag{2}
   \]

   c) Show that one solution of this equation can be obtained by setting \(\frac{\partial^2 \delta}{\partial x^2} = 0\), and that this solution matches into the growing mode in the matter dominated era, when \(x \gg 1\).

   d) Using the solution in part c), show that the scale invariant spectrum of gravitational potential fluctuations gets suppressed by \(k^4\) at high frequencies due to suppressed growth in the radiation era. What’s the physical interpretation of the characteristic wavenumber below which the spectrum shape reflects its primordial (scale-invariant) shape?