Constraints on inflation from the CMB

Why do we need inflation?

1) Flatness problem: why \( P_{\text{tot}} \) is so close to \( P_{\text{flat}} \):
   \[ P_{\text{tot}} = (1.02 \pm 0.02) P_{\text{flat}}. \]

2) Horizon problem: looking at CMB we should expect regions of only 1 sq degree to be in the same temperature, yet we see the same temperature all over the sky in accuracy of \( 10^{-5} \).

3) Particle physics predict many relics like monopoles, domain walls etc at high enough number density to overclose the Universe.

So we need a period of accelerated expansion that is also long enough.

4) Also inflation provides a mechanism for generating primordial perturbations, on a smooth universe. (Quantum perturbations are produced on scales within the horizon, get redshifted until exiting the horizon during inflation, and when they re-enter at RAD or MAT period they provide the primordial perturbations).

Suppose now that the dynamics of inflation are dominated by a scalar field \( \phi \).

Then
\[ T^{\mu \nu} = \nabla^\mu \phi \nabla_\nu \phi - L g^{\mu \nu}, \]
with
\[ L = \frac{1}{2} (\nabla^2 \phi)^2 - V(\phi) \]

Taking \( T^{\mu \nu} = 0 \) we get the eq of motion:
\[ \ddot{\phi} + 3H \dot{\phi} + V'(\phi) = 0 \]
where \( \dot{\phi} = \frac{d\phi}{d\tau} \)

During inflation the scale factor increases exponentially.
Defining \( N \) the number of e-folds before the end of inflation at which a mode crossed the horizon, then

\[
N = \int \frac{H dt}{\dot{\phi}} = \int \frac{\dot{\phi}}{\phi} \frac{d\phi}{\sqrt{2\Phi V(\Phi)}}
\]

\( \Phi \) denotes the end of inflation.

Assuming slow roll,

\[
\frac{\ddot{\phi}}{\dot{\phi}} \ll \frac{\dot{\phi}^2}{2} \ll V(\phi)
\]

So from the eq. of motion we get that

\[
\ddot{\phi} = -\frac{V(\phi)}{3H^2}
\]

Also, \( \rho = V(\phi) = P \)

and the Friedmann eq.: \( H^2 = \frac{8\pi G}{3} V(\phi) \)

Also we have to have: \( V'' \ll V' \ll V \)

Defining the slow roll parameters:

\[
\epsilon = \frac{\mpl^2}{4\pi} \left( \frac{H(\Phi)}{\dot{\phi}} \right)^2 \approx \frac{\mpl^2}{16\pi} \left( \frac{V(\phi)}{V(\phi)} \right)^2
\]

\[
\eta = \frac{\mpl^2}{4\pi} \frac{H''(\Phi)}{H(\Phi)} \approx \frac{\mpl^2}{8\pi} \left[ \frac{V''(\phi)}{V(\phi)} - \frac{1}{2} \left( \frac{V'(\phi)}{V(\phi)} \right)^2 \right]
\]

where the first expressions are the definitions while the right expressions are lowest order approximations.

As \( V', V'' \ll V \Rightarrow \epsilon, \eta \ll 1 \). Using the Friedmann eq. we can get \( \frac{\ddot{a}}{a} = H^2(1-\epsilon(\phi)) \), so inflation ends at \( \epsilon(\phi) \approx 1 \).

As we have said inflation provides a mechanism to generate metric perturbations. These generated during inflation can be either scalar (compaction) or tensor (gravitational wave) perturbations. The initial conditions for super-horizon and non-linearly perturbations can be calculated back from the data.
To describe the power spectrum of the comoving curvature perturbations, $P_R$ ($= K^3 P_S$) we use the spectral index $n_R$ $P_R \propto k^{n-1}$. For $n=1$ we have a scale invariant power spectrum. Also we need $r = \frac{P_T}{P_R}$ (tensor to scalar ratio).

In the limit of slow roll $\epsilon$ and $\eta$ can be written (to first order in slow roll parameters) as:

$$n = 1 - 4\epsilon + 2\eta$$

and $r = 16\epsilon$.

So knowing $P_R$, $P_T$ and $n$, we (to first order in $\epsilon$, $\eta$) can describe the various models. Once including higher order effects we include also the $d\eta/d\epsilon$ to describe the summing of the values of $n$.

So for a specific model from $V_S$ we can calculate $\epsilon$, $\eta$ $\rightarrow r$, $n$, $N$, (to first order).

The single-field inflationary models can be classified into 3 general classes: large field, small field and hybrid (linear models are in the boundary between large and small field models).

Even though one can create models that don't fit any of these classes, also going beyond first order in $\epsilon$ and $\eta$ models can evolve from one region of $r$-$n$ to an other.

Large-field models are characterized by $V''(\phi) > 0$ and $-\epsilon < \eta < \epsilon$. In this category belong polynomial potentials $V(\phi) = \lambda^4 (\phi)^p$ and exponential ones $V(\phi) = \lambda^4 \exp(\phi/\mu)$.

For $V(\phi) \propto \phi^p$ we can get eventually $r = \frac{8}{p+2} (1-n)$ while for $V(\phi) \propto e^\phi$ we get that:

$$r = 8 (1-\eta)$$
Small-field models are characterized by $V''(\phi) < 0$ and $\mu < -3$. A generic form of potential in this category is $V(\phi) = \lambda^4(1 - \frac{\phi^p}{p})^r$. Taking $p=2$ we get $r = \frac{1 (1-n)}{2 (1-n) \exp(-1 - N(1-n))}.$

The linear models $V''(\phi) \propto \phi$ lying in the boundary of the two above classes have $V''(\phi) = 0$ and $\mu = E$. In this case $r = \frac{3}{2} (1-n).$

Hybrid models are characterized by $V''(\phi) > 0$, $0 < E < \mu$. Potentials of the form (for the inflation field $\varphi$) $V_\varphi = \lambda^4(1 + \frac{\varphi^p}{p})^r$ belong in this category. This class gets a large region in $r-n$ space due to freedom of having the second field that is responsible for ending inflation.

Results:

We will show plots of likelihood contours on $r-n$ planes.

We will present results based on CMB data only taken from WMAP 3-year run, and results where extra constraints were applied from the SDSS constraints on the power spectrum of galaxy clustering taking also $H_0 = 100h$ km/s/Mpc with $h = 0.72 \pm 0.08$. See Fig. 1 from astro-ph/0605338v1 (paper 1).

No running of the spectral index $\frac{d\ln n}{d\ln k} = 0$.

Taking $\frac{d\ln n}{d\ln k} = 0$ and using only WMAP 3-year data we get at 95% C.L. that $0.4 < n < 1.04$ and $r < 0.60$. So models that predict $n<0.30$ are ruled out at high significance. Also if we take the constraint $n<1.0$ (negative tilt) we have $r<0.4$ within 95% C.L. Including the SDSS data results in stronger constraints namely $0.35 < n<1.0$ and $r<0.31$ at 95% C.L.

One thing to notice is that the Harrison-Zel'dovich model $n=1, r=0$, $\frac{d\ln n}{d\ln k} = 0$ is within the allowed region in both cases. Still there are other groups that suggest $n=2$, being already excluded (see Fig. 4 of the paper 1 and Fig. 5 of paper 2).

Including running of the spectral index \( \frac{\text{d}n}{\text{d} \ln k} \neq 0 \) higher values of \( n \) and \( r \) are achieved. (See Fig 2 and Fig 3 from paper 1). For instance once combining WMAP 3 and SDSS we get \( 0.87 < n < 1.2 \), \( r < 0.38 \), \(-0.13 < \frac{\text{d}n}{\text{d} \ln k} < 0.07 \). Notice that in any case \( \frac{\text{d}n}{\text{d} \ln k} = 0 \) is not excluded at more than 95% C.L.

In Fig 4 from paper 1 see \( 2 \Phi^4 \) vs \( \frac{1}{2} m^2 \Phi^2 \) how they compare to the results.

Equal notes.

Some extra information from WMAP 5 year data

We will compare these data to specific inflationary models. For potential of the form \( V(\Phi) \propto \Phi^4 \) (referred also earlier) \( r = \frac{4p}{N} \) and \( 1-n = \frac{p^2}{2N} \). \( N \) again # of e-folds left...

From Fig 5 (paper 2) you see that models with \( V(\Phi) \propto \Phi^4 \) get excluded even with \( N=60 \) at 95% C.L. While models of \( V(\Phi) \propto m^2 \Phi^2 \) are well within 95% C.L. even for \( N=50 \). Also these are models with many axion fields. N-flation models (\( N \) now stands for the # of the fields). Those models give spectral indices that are further away from 1 compared to the 1-field with \( V(\Phi) \propto m^2 \Phi^2 \) but predict the same tensor/slow ratio, \( r \). For \( N>50 \) models fall within the 95% C.L.

For the case of exponential potential \( V(\Phi) \propto \exp\left[ -\frac{\Phi^2}{2m^2} \right] \) \( r \) and \( n \) are given by \( r = \frac{16}{P} \), \( 1-n = \frac{2}{P} \). Again from Fig 5 (paper 2) models with \( P < 60 \) are excluded at more than 95% C.L. models with \( P > 70 \) are within 95% while models with \( P = 120 \) are at 68% C.L. Note: Instead of having one single field with \( P \) as mentioned above \( i.e. P = 120 \) we can have \( N \) # of fields each with \( P = P_i \) and in that case \( N \cdot P = 120 \).

For hybrid models of the form \( V(\Phi) = V_0 + \frac{1}{2} m^2 \Phi^2 + m \Phi \) we define \( \Phi = \frac{m \Phi}{V_0} \) to describe which of the \( V_0 \) or \( \frac{1}{2} m^2 \Phi^2 \) term...
is the dominant one. So depending on the value of $\Phi$ that corresponds to the field value at which a wavelength that we probe left the horizon we can separate r-n. par. space into regions (there are of course models that move from one to an other region).

When $\Phi \leq 1$ ($V = V_0$) we are at the flat potential regime, while for $\Phi >> 1$ ($V = \frac{1}{2} \mu^2 \phi^2$) we are back at case of large field chaotic regime. For $\Phi \sim 1$ we're both terms in the potential are significant we are at the transition regime ($\frac{2}{3} \leq \Phi \leq 1$). From WMAP 5-year data (see Fig 5 of paper2) the flat DH regime is not within 95% CL, the transition regime is outside the 68% but inside the 95% and the chaotic regime contains the 68% CL regime.