Primordial Non-Gaussianity from Single Field Inflation

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In single field inflationary model one considers the action of a single scalar field coupled to gravity. This action, when expanded around the homogeneous background, gives terms that are quadratic in the small fluctuations in leading order. In other words, we have a free field theory, which is a collection of harmonic oscillators. The quantum fluctuations are in the ground state (because otherwise perturbations themselves have too much energy and \( p = -p \) does not hold), so that to leading order we have Gaussian spectrum of primordial fluctuations, which is also scale invariant.

But the full theory is not free, and we will have small deviations from Gaussianity coming from the (cubic) interaction terms that arise from the nonlinearity of the Einstein-Hilbert action as well as self-interaction of the inflaton. Physical observables such as, the three-point function of scalar fluctuations, or its Fourier transform, the bispectrum can tell apart non-Gaussian from Gaussian spectrum for which any odd-point function vanishes. (Note that calculation of}
As usual our "scale" fluctuation is the famous $\xi$ defined as:

$$\xi = -\Phi_H - \frac{\alpha H}{k} \nu ; \quad \Phi_H = \text{Bardeen potential}, \nu = \text{stress-energy variable}$$

The virtue of this gauge-invariant quantity is two-fold: First, in spatially flat slicing it is proportional to inflaton fluctuation, while in conformal Newtonian gauge it is proportional to Newtonian potential:

$$\xi = \begin{cases} 
-\frac{a H}{\dot{\Phi}_0} \delta \phi & \text{(Spatially flat slicing)} \\
-\frac{5 + 3 \omega}{3 + 3 \omega} \Phi & \text{(Conformal Newtonian gauge)}
\end{cases}$$

Second, $\xi$ is conserved at super-horizon scale. This can be seen from conservation of stress-energy tensor for adiabatic perturbations, which single-field inflation generates. Therefore $\xi$ is very useful because after the end of inflation inflaton ceases to determine the dynamics of the universe, and only the mode reenters the horizon it is affected by $\Phi$. Note that in comoving gauge $\xi = -\Phi_H$, and it is clear that $\xi$ is a curvature perturbation. We see that

$$P_\xi(k) = \left( \frac{a H}{\dot{\Phi}_0} \right)^2 P_{\delta \phi} = \frac{4 \pi G}{c} P_{\delta \phi} = \frac{4 \pi G}{c} \left( \frac{H^2}{\delta k^2} \right)$$

$$E = \frac{1}{2} \kappa m^2 \left( \frac{\nu'}{\nu} \right)^2$$

Therefore the primordial two-point function is:

$$\left\langle \xi_{-\mathbf{k}_1} \xi_{-\mathbf{k}_2} \right\rangle = \left[ \frac{2 \pi}{k_1 + k_2} \right] \frac{4 \pi G}{c} \frac{H^2}{\delta k^2} \left( \frac{\nu'}{\nu} \right)^2$$
We are instead interested in the 3-point function: \( \langle 5_{\eta_1}, 5_{\eta_2}, 5_{\eta_3} \rangle \).

In the limit \( k_3 \ll k_1, k_2 \), there is a simple argument that determines the 3-point function. Since \( k_3 \) is much smaller than \( k_1, k_2 \), it crosses the horizon during inflation much earlier than \( k_1, k_2 \). By the time \( k_1, k_2 \) crosses the horizon, \( 5_{\eta_3} \) is constant!

So the only effect \( 5_{\eta_3} \) can have on \( 5_{\eta_1} \) and \( 5_{\eta_2} \) is through the fact that a non-zero \( 5_{\eta_3} \) is a slight change in density of the universe, \( a \) is curvature, or local scale factor. In effect, the \( 5_{\eta_3} \) fluctuation will rescale the other two momenta, so that now they cross the horizon at a slightly earlier time. Therefore, one gets a contribution proportional to the violation of scale-invariance of the two-point function \( \langle 5_{\eta_1}, 5_{\eta_2} \rangle \):

\[
\langle 5_{\eta_1}, 5_{\eta_2}, 5_{\eta_3} \rangle = - \left( \langle 5_{\eta_1}, 5_{\eta_2} \rangle \frac{d}{dk} \langle 5_{\eta_3} \rangle \right)
\]

\[
= - (n_3 - 1) \langle 5_{\eta_1}, 5_{\eta_2} \rangle \langle 5_{\eta_3} \rangle
\]

where \( n_3 \) is the tilt of the scalar spectrum given by \( \langle 5_{\eta_3} \rangle \sim h^{n_3 + 1} \).

Due to momentum conservation, the other possibility is that all the \( k_i \)'s are the same order of magnitude. To compute this one considers the cubic terms in the Lagrangian. These cubic terms lead to a change in the vacuum state of the fluctuations as well as non-linearities.
in the evolution. One needs to work in second order perturbation theory to obtain an expression for 3-point function. Therefore the correct non-linear generalization of $\xi$ is also required. Besides, there are gauge issues which must be taken care of. One gets:

$$
\langle \xi_{k_1} \xi_{k_2} \xi_{k_3} \rangle = (2\pi)^3 \delta^3(k) \left( \frac{\dot{H}}{\dot{\phi}} \right)^4 \left[ \frac{1}{1!(2\Pi)^3} \left\{ (2e-2\eta) \sum k_i^3 + \sum \epsilon_{ij} k_i k_j + 8 \epsilon \sum \frac{k_i k_j}{k} \right\} \right]
$$

momentum conservation amplitude $\sim 10^{-8}$ \hspace{1cm} function of degree $-6$

The result is first order in slow roll parameters. The term in the square brackets is a homogeneous function of degree $-6$ by approximate scale invariance. Note that $\chi^m$ does not show up; in fact it comes in $O(\epsilon^2)$. That is leading interaction is coming from gravity, not from the potential.

But we see how big this non-Gaussianity is. One interesting quantity is the zero angular separation limit, the skewness:

$$
\frac{\langle \xi_{k_1} \xi_{k_2} \xi_{k_3} \rangle}{\langle \xi^2 \rangle^{3/2}} \sim \frac{H^2}{\dot{\phi}} O(\epsilon, \eta) \sim 10^{-7}
$$

Therefore, the theory is tremendously weakly coupled, and it is very close to Gaussian. This makes it very hard to be potentially measurable.