Homework Set #3  (Due 11/16 in Class)

1. The “horizon problem”.

a) Show that in the standard Big Bang model the horizon distance today is to a very good approximation \( d_H \approx 2H^{-1} \), even though the universe changed from radiation to matter dominated sometime in the past. Use reasonable values for \( t_{eq}/t_0 \). At what redshift does this approximation become better than 10%?

b) Consider now the evolution of \( d_H \) and \( H^{-1} \) when there is a period of inflation. Assume the universe is radiation dominated before and after the period of inflation, which lasts 60 e-folds. Show that the horizon and the Hubble radius are very different today. Explain why the behavior of \( d_H \) solves the “horizon problem”. Sketch a graph of scales, \( H^{-1} \), and \( d_H \) as a function of scale factor.

c) Show that once scales are in causal contact (as measured by \( d_H \)) they stay forever in causal contact.

d) The explanation in b) is not the usual explanation found in discussions about inflation (e.g. in books), which actually involves \( H^{-1} \) rather than \( d_H \). Why is this so, and why does the “standard” explanation make sense?

2. Gaussian Random Fields and Vacuum Fluctuations

a) Show that for a Gaussian probability distribution function (PDF), \( P_G(\phi) \equiv (2\pi\sigma^2)^{-1/2} \exp[-\phi^2/(2\sigma^2)] \), the moments are given by \( \langle \phi^{2n} \rangle = (2n-1)!! \langle \phi^2 \rangle^n \), where \( \langle \phi^2 \rangle = \sigma^2 \) and \( (2n-1)!! = (2n-1)(2n-3) \ldots 1 \).

b) All the moments can be generated from the moment generating function \( M(\lambda) \equiv \langle \exp(\lambda\phi) \rangle \), i.e. \( \langle \phi^n \rangle = (d^n M(\lambda)/d\lambda^n)_{\lambda=0} \). Using the result in a), show that for a Gaussian field \( M(\lambda) = \exp[\lambda^2\sigma^2/2] \).

c) Show that vacuum quantum fluctuations obey Gaussian statistics. 

\textit{Hint:} Expand each mode \( \phi_k = w_k a_k + w_k^* a_k^\dagger \) in terms of creation and annihilation operators \( a_k^\dagger \) and \( a_k \), obeying the usual commutation relations and calculate the moment generating function for such a mode. For operators \( A \) and \( B \) that commute with their commutator \([A, B]\), the following may come handy: \( e^{A+B} = e^A e^B e^{-\frac{1}{2}[A, B]} \).
3. Fluctuations from Inflation.

a) Consider the model of chaotic inflation, where the potential is given by $V(\phi) = V_0 \phi^p$, with $p \geq 2$. Calculate the slow-roll parameters $\epsilon(\phi)$ and $\eta(\phi)$ in terms of parameters of the potential and find the amplitude of the field when inflation ends (justify your prescription here) and 50 e-folds before the end of inflation.

b) Assuming that the amplitude of fluctuations for modes that become larger than $H^{-1}$ 50 e-folds before the end of inflation is $\Delta(k) \sim 10^{-11}$ (as observed by the COBE and WMAP satellites), calculate the energy scale of inflation, the scalar spectral index $n_S$ and the tensor spectral index $n_T$. How important is the contribution of tensor modes compared to scalar modes when $p = 4$?

c) Assuming the slow-roll approximation, find the most general form of the inflaton potential so that scalar perturbations are exactly scale-invariant.