We discussed the Schwarzschild geometry already but concentrated on effects observed when $r > r_s = 2M$. This is because for normal stars their radius is so much larger than $r_s$, that the region $r < r_s$ is not accessible (the Schwarzschild metric corresponds to the metric outside a spherical body). For example, for the sun $r_s \approx 3$ km while the solar radius is $R_\odot \approx 7 \times 10^5$ km, so the closest we can get before we hit the surface is $r \approx 2 \times 10^5 r_s$. Inside the sun one must solve for the metric in presence of matter (a very different solution).

The sun is in equilibrium between radiation pressure (that results from thermonuclear fusion releasing energy in the form of radiation) and gravity. There are other types of stars in which pressure is supplied by a different mechanism: white dwarfs and neutron stars. Both are the result of the end of stars lives, when they run out of fuel to make fusion energetically favorable.

Normal stars function by “burning” (fusion) hydrogen into helium, then into carbon, oxygen etc., building heavier elements as they go along, which have more binding energy and thus energy can be extracted out of that (which supplies the radiation pressure). However, this cannot continue forever, since elements in the iron group are the most bound, i.e., one cannot fuse them into something that will release energy. As a result gravity pulls the star inwards until a new source of pressure takes over: this is supplied by quantum mechanical degeneracy pressure of fermions due to Pauli's principle.
White dwarfs correspond to the case when the pressure is supplied by degeneracy pressure from the electrons. In this case, a star of order solar mass ($\sim M_\odot$) can reach a radius of order a few $10^3$ km, i.e., two orders of magnitude smaller than the Sun. Therefore, one can probe the Schwarzschild metric down to $\frac{S\text{tor}}{r_{\text{S}}} \approx 10^3$, still viable compared to $r_\text{S}$.

For stars sufficiently massive (say $\sim 10 M_\odot$), the degeneracy pressure of electrons is not enough to counteract gravity and the star continues to collapse until the degeneracy pressure of neutrons takes over (protons combine with electrons into neutrons by inverse beta-decay).

Typical sizes of neutron stars are of order 10 km, so $\frac{S\text{tor}}{r_{\text{S}}}$ is few and we can probe distances very close to $r_{\text{S}}$ before we hit the surface!

However, for stars massive enough (>$\sim$ few $M_\odot$), neutron degeneracy pressure is not enough to stop the star from collapsing, and we know of no other pressure source that will take over at high-enough densities. In these cases, we think that stars collapse into a singularity (corresponding to $r=0$ in the Schwarzschild solution), and that the Schwarzschild metric describes the spacetime everywhere $r>0$. This is the Schwarzschild black hole.

In Schwarzschild coordinates the metric reads

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta\, d\phi^2)$$

It is clear that something strange is going on at $r=2M=r_\text{S}$, where $g_{tt} \to 0$, $g_{rr} \to \infty$, and they change sign as $r<2M$. What is not clear is whether the singularity in the metric is physical...
or whether it is just a coordinate singularity, i.e.
problems with the choice of coordinate themselves.
We can suspect is the latter because when we considered
radial plunge geodesics (l=0) we saw that measured in
proper time, there is nothing strange going on at \(r = 2M\) for
such a falling observer, as we found that (\(2 \leq r \leq 4\))

\[
\Gamma(t) = \left(3/2\right)^{1/3} r^{1/3} \left((2/3) \frac{2}{2M} \right)^{2/3}
\]

while if we changed from \(t\) to \(t'\) we saw that
it took \(\infty\) time to reach \(r = 2M\). This suggests there is
a problem with the time coordinate near \(r = 2M\).
To see that
the problem is indeed a coordinate singularity, it suffices to
show a single coordinate system in which \(r = 2M\) is not
singular. The simplest one is to take a redefinition of

\[
t = r - r - 2M \ln \left| \frac{r}{2M} - 1 \right|
\]

(\text{Eddington-Finkelstein})
coordinates

after which the metric reads (for \(r > 2M\) or \(r > 2M\))

\[
ds^2 = - \left(1 - \frac{2M}{r}\right) \, dt^2 + 2 \, dv \, dr + r^2 \left( dr^2 + \sin^2 \theta \, d\phi^2 \right)
\]

Note this is the same geometry, only in different coordinates. We see
no element of the metric diverging at \(r = 2M\) now, so nothing
singular is going on there, as expected from the earlier
geodesic argument given above. The metric is now not diagonal,
but that's a small price to pay for a non-singular description
of spacetime. Note however that \(r = 0\) continues to be
a singularity of the metric in this new coordinate system (and
all coordinate systems): this is a physical singularity.

\[\text{Note: why (*)? Radial geodesic of incoming photon obeys } \frac{ds}{dr} = -(1-2M)^{-1}\]
\[\Rightarrow t = -r - 2M \ln \left| \frac{r}{2M} - 1 \right| + \text{cont.} \quad \text{so } r \text{ is that constant!}\]
Let's now look at light rays in Eddington coordinates. Take radial geodesics $d\theta - d\phi = 0$. Since $ds^2 = 0$ we have

$$-(1 - \frac{2M}{r}) \, dv^2 + 2dv \, dr = 0$$

One possibility is that $(r \geq 2M)$ \[ dv = 0 \]

As expected, $v = \text{const.}$ describes incoming photons - hence $v$ is a null coordinate.

Which corresponds to ingoing radial light rays because

$$v = t - r + 2M \ln \left( \frac{r}{2M} - 1 \right)$$

So to keep $v = \text{const.}$ not in terms of $r$ must decrease $\Rightarrow$ ingoing.

Another possible solution $(r \geq 2M)$ is that instead,

$$-(1 - \frac{2M}{r}) \, dv + 2dr = 0$$

or, integrating,

$$v - 2 \left( t + 2M \ln \left( \frac{r}{2M} - 1 \right) \right) = \text{const.}$$

or

$$v - 2 \left( t - r + 2M \ln \left( \frac{r}{2M} - 1 \right) \right) = \text{const.}$$

for $r \geq 2M$; as $r \to 2M$ \[ t - r \to \text{const.} \Rightarrow t = r + \text{const.} \Rightarrow \text{outgoing} \]

for $r \leq 2M$; will be ingoing (see plot below)

Finally, consider what happens at $r = 2M$. Then the null condition says

$$dv \, dr = 0$$

Which means $r = 2M$ is a solution for a null surface ($dr = 0$, at all times $v$).

We can put all these results together by plotting these radial geodesics for photons as a function of time coordinate.

$$\tau = v - r = t + 2M \ln \left( \frac{r}{2M} - 1 \right)$$

This makes sense, rather it corresponds more using $\tau$, since $v$ is a null coordinate, whereas $t$ is time-like.
which makes the ingoing \( v = \text{const.} \) light rays a 45° line in the \( t-r \) plane:

\[ v-2 (r+2M \ln \left( \frac{r}{2M} - 1 \right)) = \text{const} \text{ for } r>2M \]

\[ v=\text{const.} \text{ (in coming light rays)} \]

As we approach \( r=2M \), the outward side of the light cone becomes vertical, meaning light cannot escape inside the event horizon.

\[ ds^2 = -\left(1-\frac{2M}{r}\right) dv^2 + 2dv dr + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]

so \( r=\text{const.} \) surfaces have \( ds^2 = A dv^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \) with \( A>0 \), so such surfaces are spacelike in every direction.

This means that the \( r \) coordinate has become timelike, and \( r=0 \) is not a plane in space but a moment in time. That is why it is irreversible that something at \( r=2M \) eventually hits \( r=0 \), i.e. \( r \) is "clocking down". Some can be seen from Schwarzschild coordinates.
Collapse to a Black Hole

With the information we have we can now describe (at least qualitatively) how a star collapses into a black hole, and how this process is seen by a stationary observer at large radius -

For simplicity, let's consider the case of a star made of "dust", i.e. matter with zero pressure - In this situation absence of pressure forces means particles on the outer surface of the star will move on radial geodesics of the Schwarzschild geometry, which we already derived.

Consider two observers: one observer participates in gravitational collapse by riding the surface of the star down to $r=0$, and the other remains fixed at large radius. The infalling observer carries a clock and communicates with the distant observer by sending light pulses at equally-spaced intervals according to this clock (which measures proper time along the infalling radial geodesic). The spacetime diagram in Eddington-Finkelstein coordinates looks like:

- horizon forms when surface of star is enclosed inside event horizon
- these outgoing photons will never be received by distant observer
- inside star geometry is not Schwarzschild
- observer on surface, falling on some radial geodesic
- at reception, $\Delta \xi$ increases as time goes on

observers
Since the metric in these coordinates is
\[ ds^2 = -(1 - \frac{2M}{r}) \ dt^2 + 4M \ dt \ dr + (1 + \frac{2M}{r}) \ dr^2 + r^2 \ (d\theta^2 + \sin^2 \theta \ d\phi^2) \]
we see that at large \( r \) \( \mathcal{E} \) is also proportional to measured by
the distant observer (and agrees with Schwarzschild time \( \tau \) as well).
From the diagram we see the distant observer measures
increasing separation between light pulses, and in fact the
last pulse received is just before the infalling observer goes
through \( r = 2M \). In particular, the light emitted towards
\( r = 2M \) stays there (horizon is a null surface) and formally reaches
the distant observer at \( \mathcal{E} = \infty \). Pulses emitted after
\( r = 2M \) do not go to larger \( r \), but rather to smaller \( r \) and eventually reach \( r = 0 \). Also, all radial (and non-radial
as well) time-like geodesics fall into \( r = 0 \) which means that once
the surface of the star goes across the Schwarzschild radius,
gravitational collapse to a singularity is inevitable: the surface
\( r = 2M \) is a closed trapped surface i.e., a closed spacelike surface such
that the area of light pulses emitted inward and outward decreases
with time. Since matter can only move slower than light, and
light emitted in both directions eventually hits \( r = 0 \), all matter must do
the same : matter is trapped between light surfaces that go to zero
area must collapse to a singularity! (This is the basis for the
famous singularity theorems in GR).

While the infalling observer collapses down to \( r = 0 \), the distant
observer only sees the star approaching \( r = 2M \) but never quite
cross the Schwarzschild radius. Furthermore, not only the light
pulses get in longer and longer intervals, but also the light
that arrives it does with highly redshifted (with redshift
going to infinity for light emitted close to $r=2M$.

Both of these effects mean that the distant observer sees the luminosity of the star approach to zero as $r \to 2M$, so it looks like a black hole of radius $r=2M$. Let's estimate the timescale for this.

An outgoing light ray obeys (see 2nd eq. in box on page 4)

$$\gamma = 2 \left( \frac{r+2M}{1-\frac{r}{2M}} \right) = \text{const.}$$

So we have at emission ($E$) and reception ($R$)

$$\gamma_E = 2 \left( \frac{r_E + 2M}{1-\frac{r_E}{2M}} \right) = \gamma_R = 2 \left( \frac{r_R + 2M}{1-\frac{r_R}{2M}} \right)$$

but when $r_E \approx 2M$ the logarithm dominates, i.e., $2M \ln \left| \frac{r_E}{2M} - 1 \right| \gg r_E, r_R$.

while for $r_R \gg 2M$ it is subdominant $\frac{r_R}{2M} \gg 1$.

Then:

$$\gamma_R = 4M \left( \ln \left| \frac{r_E}{2M} - 1 \right| + \frac{r_E - r_R}{2M} \right) \approx \gamma_R - 2 \gamma_R + \frac{r_E - r_R}{2M}$$

$$\approx \gamma_E - 2M \left[ 1 + e^{-\frac{(r_E - r_R)}{4M}} \right]$$

so as $r_E$ increases, $\gamma_E$ approaches $2M$ exponentially with a characteristic timescale $4M$.

In time units,

$$4M = \frac{4 \cdot 0.5 M}{c^2} = 2 \times 10^{-5} \left( \frac{M}{M_0} \right) \text{ seconds}$$

so approach to a black hole is really fast!

\textbf{Eddington-Finkelstein Coordinates}

In Eddington-Finkelstein coordinates, while the incoming photon worldline is a simple $4\xi$ line in $\xi, r$ plane, the outgoing photon worldlines are complicated, and in fact, behavior changes drastically in a
A nontrivial way as \( r = 2M \) is crossed (which is physically expected as \( r = 2M \) outgoing directed light becomes in coming).

The question arises whether it is possible to find a coordinate system where both incoming and outgoing radial photon geodesics are continuous straight lines: the answer are the KS coordinates \( U \) and \( V \) deformed from Schwarzschild \( t, r \) as:

\[
\begin{align*}
U &= \left( \frac{r}{2M} - 1 \right)^{1/2} e^{r/4M} \cosh t/4M \quad r > 2M \\
V &= \left( \frac{r}{2M} - 1 \right)^{1/2} e^{r/4M} \sinh t/4M
\end{align*}
\]

which leads to the following metric (valid for all \( r \))

\[
d\mathbf{s}^2 = \frac{32M^3}{r} \frac{r/2M}{e} \left( -dU^2 + dV^2 \right) + r^2 \left( d\phi^2 + \sin^2 \phi \, d\theta^2 \right)
\]

where we still have to express \( r \) in terms of \( U, V \) according to,

\[
\left( \frac{r}{2M} - 1 \right) \frac{r/2M}{e} = U^2 - V^2 \quad (\#)\]

Again you can see that the Schwarzschild solution is not singular in those coordinates, showing once again that the singularity in Schwarzschild coordinates for \( r = 2M \) was not physical.

We can now make plots of geodesics in the \( U, V \) plane known as the Kruskal diagram. Light propagation along radial geodesics (\( d\phi = 0 \)) are straightforward:

\[
dU^2 = dV^2 \quad \Rightarrow \quad U = \pm V
\]

Thus light cones are everywhere \( \pm 45^\circ \) lines!
From eq. (x) we see that \( r = \text{const.} \) corresponds to \[ U^2 - V^2 = \text{const.} \]

i.e. hyperbolas in the \( U, V \) plane. Whether the const. is positive or negative depends on the value of \( r \) (\( r > 2M \) and \( r < 2M \) respectively).

In particular, \( r = 0 \) gives

\[ U^2 - V^2 = -1 \implies V = \pm \sqrt{U^2 + 1} \]

So far we have

\[ \begin{cases} \frac{V}{U} = \tanh \left( \frac{t}{4M} \right) & \text{if } t > 2M \\ \frac{U}{V} = \tanh \left( \frac{t}{4M} \right) & \text{if } t < 2M \end{cases} \]

This gives

\[ t = \pm \infty \quad U = V \]

Thus they correspond to straight lines in the \( UV \) plane.

\[ t = 0 \text{ gives} \]

\[ V = 0 \quad r > 2M \]

\[ U = 0 \quad r < 2M \]
We see then that the whole EF coordinate space with \(-\infty < V < +\infty, 0 < r < \infty\) is mapped into the unshaded UV plane, with \(V > U\) (the other half corresponds to a "white hole" rather than a black hole).

Note that in the metric there is no dependence of the signature on position, so always \(V\) is timelike and \(U\) spacelike, unlike \(t, r, \theta, \phi\) in Schwarzschild coordinates.

The collapsing star in these coordinates looks very simple:

![Diagram](image_url)