Evolution of the Scale Factor

To complete the description of the metric, we need to solve for the dynamics, i.e., the time-evolution of the scale factor $a(t)$, the only time-dependent part of the metric. An equation for $a(t)$ follows from Einstein equations, which give

$$(\ddot{a}/a) = \frac{8\pi G}{3} \rho$$

for the spatially flat case (we'll add curvature shortly). This is known as the Friedmann equation, and can be thought of as conservation of energy: kinetic energy of expansion is balanced by gravitational attraction, as we'll discuss in more detail below. Here $\rho$ is the total energy density (matter, radiation, vacuum, etc.), and we can rewrite this introducing the Hubble constant

$$H^2 = (\ddot{a}/a) = \frac{8\pi G}{3} \rho$$

It is common to introduce relative densities of each component in the universe by defining

$$\Omega_i \equiv \frac{\rho_i(t_0)}{\rho_{\text{crit}}(t_0)}$$

where $\rho_{\text{crit}}(t_0)$ has the value

$$\frac{8\pi G}{3} \rho_{\text{crit}}(t_0) = H_0^2 \Rightarrow \rho_{\text{crit}}(t_0) = \frac{2H_0^2}{8\pi G} \approx 1.878 \times 10^{-29} \text{ kg m}^{-3}$$

we'll see why this merits the name critical below.

Since

$$\sum \Omega_i = \sum \frac{\rho_i(t_0)}{\rho_{\text{crit}}(t_0)} = \frac{\rho(t_0)}{\rho_{\text{crit}}(t_0)} = \frac{3H_0^2}{8\pi G} \approx 1$$

$$\frac{8\pi}{3m^3} h^2$$
$1 = \sum_{i} \omega_i = \omega_{\text{mat}} + \omega_{\text{rad}} + \omega_{\text{vac}} + \ldots$

So we can write, since \(a(t_0) = 1\)

$P_{\text{mat}} \propto a^{-3}$

$P_{\text{mat}} = \frac{P_{\text{mat}}(t_0)}{a^3} = \frac{\Omega_{\text{mat}}(t_0)}{a^3}$

$P_{\text{rad}} = \frac{\omega_{\text{rad}} P_{\text{rad}}(t_0)}{a^4}$

It is convenient to take \(a(t_0) = 1\), clearly this can be done as the overall value of \(a\) has no physical significance.

We can then write

$P_{\text{tot}}(t_0) = P_{\text{mat}}(t_0) \left[ \frac{\omega_{\text{mat}}}{a^3} + \frac{\omega_{\text{rad}}}{a^4} + \omega_{\text{vac}} \right]$.

We can now go back to the Friedman equation and write,

Similarly:

$\dot{a}^2 = \frac{8 \pi G}{3} P_{\text{tot}}(t_0) \left[ \frac{\omega_{\text{mat}}}{a^3} + \frac{\omega_{\text{rad}}}{a^4} + \omega_{\text{vac}} a^2 \right] a^2$

or

$\frac{1}{2} \frac{\dot{a}^2}{H_0^2} = \frac{1}{2} \left[ \frac{\omega_{\text{mat}}}{a} + \frac{\omega_{\text{rad}}}{a^2} + \omega_{\text{vac}} a^2 \right]$.

or

$\frac{1}{2} \frac{\dot{a}^2}{H_0^2} + V_{\text{eff}}(a) = 0$

"Energy-like" for flat models

where

$V_{\text{eff}}(a) = -\frac{1}{2} \left[ -\omega_{\text{vac}} a^2 + \frac{\omega_{\text{mat}}}{a} + \frac{\omega_{\text{rad}}}{a^2} \right]$

is the effective potential for the dynamics of \(a(t)\).

This form of Friedman eqn is easy to solve for a single component:

1) Matter dominated: \(2 = \omega_{\text{mat}} = 1\) i.e. \(\omega_{\text{rad}} = \omega_{\text{vac}} = 0\)
\[ \frac{1}{2} \frac{a^2}{H_0^2} = \frac{1}{2} a \Rightarrow \dot{a} = \frac{H_0}{a^{1/2}} \]

or \[ \sqrt{a} \, da = H_0 \, dt \Rightarrow a^{3/2} \propto t \Rightarrow a \propto t^{1/2} \]

\[ a(t) = \left( \frac{t}{t_0} \right)^{1/2} \]

\( \hat{\omega} \)

ii) Radiation dominated: \[ \hat{\omega} = \hat{\omega}_{\text{rad}} = 1 \quad \hat{\omega}_{\text{mat}} = \hat{\omega}_{\text{vac}} = 0 \]

\[ \frac{1}{2} \frac{a^2}{H_0^2} = \frac{1}{2} \frac{\hat{\omega}_{\text{rad}}}{a^2} = \frac{1}{2} a^2 \Rightarrow \dot{a}^2 = \frac{H_0}{a} \]

\[ a \, da = H_0 \, dt \Rightarrow a^2 \propto t \Rightarrow a \propto t^{1/2} \]

\[ a(t) = \left( \frac{t}{t_0} \right)^{1/2} \]

\( \hat{\omega} \)

iii) Vacuum dominated: \[ \hat{\omega} = -\hat{\omega}_{\text{vac}} = -1 \]

\[ \frac{1}{2} \frac{a^2}{H_0^2} = \frac{1}{2} a^2 \Rightarrow \dot{a} = a H_0 \Rightarrow a(t) = a H_0 (t - t_0) \]

Note in this case \[ H^2 = \frac{8 \pi G}{3} \hat{\omega}_{\text{vac}} = \frac{8 \pi G}{3} \hat{\omega}_{\text{vac}} (\text{const.}) = H_0^2 \]

\( \Rightarrow H = \text{const.} \quad \text{the universe expands exponentially.} \)

Note, that unlike previous case, where \( \rho \to 0 \) as \( t \to 0 \), here \( \rho_{\text{vac}} = \text{const.} \) and it does not diverge as \( t \to 0 \) (no singularity).

Since \( \rho_{\text{rad}} \) peaked early on, then \( \rho_{\text{mat}} \), and at late times for \( \rho_{\text{vac}} \) (see graph last class) we can roughly plot the \( a(t) \) as

\[ a(t) = \begin{cases} \propto t^{1/2} & \text{for } a \to a_{\text{rad}} \\ \propto t & \text{for } a \to a_{\text{mat}} \\ \propto t^{3/2} & \text{for } a \to a_{\text{vac}} \end{cases} \]
We see that in MAT & RCD, the scale factor goes to zero as \( t \to 0 \) and the density diverges at \( a = 0 \), thus is the big bang singularity (we'll see later that curvature blows up there too, so it is a physical singularity). Note \( p \) diverges everywhere at \( a = 0 \), not at a specific point in space, so the singularity is not a place in space but rather a moment in time (which we assign to \( t = 0 \)) - In practice as we extrapolate back from \( t \to \infty \), \( p \) becomes so high that the laws of physics (including Einstein's gravity) break down, so we cannot really trust extrapolating all the way back to \( t = 0 \).

Let's take for example a universe dominated by MAT all the time, then from \( \Omega = (t/t_0)^{2/3} \) we can calculate its age in terms of \( t_0 \):

\[
\dot{a} = \frac{2}{3} \left( \frac{t}{t_0} \right)^{-1/3} \frac{1}{H_0} \\
\frac{t_0}{t} = \frac{3}{2} \left( \frac{a_0}{a} \right)^{-1/3} \quad \Rightarrow \quad t_0 = \frac{2}{3} \frac{a_0}{H_0} \frac{1}{\sqrt{\Omega}} = \frac{2}{3} H_0
\]

and since \( H_0 = t_0 \) is Hubble time \( \sim 13 \) Gyr \( \gg t_0 \sim 9.6 \) yr. Which is much smaller than the age of oldest stars (in globular clusters) in our own galaxy! This points to the fact that the universe cannot be MAT-dominated (including RCD does not help, one needs vacuum energy as well) -

Option: Note that from the metric for spatially flat (h=0) RW one we see that the volume of \( t = \text{const} \) spacelike surfaces is infinite, however the observable universe (how much we can see from signals that propagate at the speed of light) is finite.
Let's compute that, essentially we are looking at the spectral extent of our past-lightcone at the big bang. Roughly, should be of order the age of the universe times the speed of light, \( cH_0 \). Since

\[
 ds^2 = -dt^2 + a^2 \left[ dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \right]
\]

it is convenient to introduce conformal time \( \eta \)

\[
 d\eta = dt
\]

\[
 ds^2 = a^2(\eta) \left[ -d\eta^2 + dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \right]
\]

so that the metric is conformally flat, i.e., Minkowski times an overall conformal factor \( a^2 \) (the word conformal means, essentially, angles are preserved). In \( \eta, r, \theta, \phi \) coordinates, things are simple as for radial light rays we have

\[
 ds^2 = 0 = a^2 \left[ -d\eta^2 + dr^2 \right] \Rightarrow d\eta = \pm dr
\]

i.e., 45° angles in a \( \eta, r \) space-time diagram.

The size of the "horizon" (largest distance from which light can reach an observer by time \( \eta \)) is

\[
 R_h(\eta) = \int_0^\eta d\eta \quad \text{or} \quad R_h(t) = \int_0^t \frac{dt}{a(t)}
\]

\( R_h(\eta) \) is the comoving horizon.
Converting to physical distance
\[ d_h(t) = a(t) r_h(t) = a(t) \int_0^t \frac{dt'}{a(t')} \]
physical
horizon
size

For current cosmology parameters \( d_h(t) \sim 14 \) Gpc.

We can easily calculate \( d_h \) for different \( a(t) \)'s given by NAT, RAD, VAC assumption:

\[
\begin{align*}
\text{NAT: } & \quad a(t) \sim t^{2/3} \quad \Rightarrow \quad d_h(t) = 3 \frac{t}{t_0} \quad (\text{NAT}) \\
\text{RAD: } & \quad a(t) \sim t^{1/2} \quad \Rightarrow \quad d_h(t) = 2 \frac{t}{t_0} \quad (\text{RAD})
\end{align*}
\]

\( t_0 \) is the present time. How come this is larger than \( t \) i.e. is light propagation speed faster than \( c \), to get 3\( c t \) in time \( t \)?

Again, if we assume NAT case, \( d_h(t_0) = 3 t_0 = \frac{3}{2} \frac{t}{t_0} = 2 + t \sim 8 \) Gpc
(much smaller than accepted value of \( \sim 14 \) Gpc, due to ignoring contribution of vacuum energy, as we'll see).

From the Friedman eqn + Energy conservation we can derive the acceleration of the expansion of the Universe:

\[
\begin{align*}
\frac{d}{da^3} & = -p \quad \Rightarrow \quad \frac{a^3}{dt} = -(\rho + p) \frac{3a^2}{t} \\
H^2 & = \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho
\end{align*}
\]

To get \( \ddot{a} \) we take \( \frac{d}{dt} \) of Friedman:

\[
\frac{2}{a} \frac{\ddot{a}}{a^2} = \dot{a} \left( \frac{\ddot{a}}{a} - H^2 \right) = \frac{8\pi G}{3} \rho = -8\pi G A (\rho + p)
\]

\[
\Rightarrow \quad \frac{\ddot{a}}{a} = H^2 - 4\pi G (\rho + p) = -\frac{4\pi G}{3} (\rho + 3p)
\]

which says that if \( \rho + 3p > 0 \) \( \Rightarrow \ddot{a} < 0 \), i.e. the universe accelerates as it expands. To get a speed up we need an equation of state with negative pressure, i.e.
if \( \rho = \text{WS} \quad \omega = \text{const.} \)

\[ \ddot{a} = -\frac{4\pi G}{3} \rho (1+3\omega) \quad \text{so need} \quad \omega < -\frac{1}{3} \quad \text{to get} \quad \dot{a} > 0 \]

so RAP or MAT does not give acceleration, but \( \omega = -1 \) (a cosmological constant, or VAC) does! The observations in fact point to current acceleration with \( \omega < -1 \) (i.e. consistent with \( \Lambda \)).

Let's take a universe with just MAT + VAC and see how the age estimate improves. We have

\[ \Omega_{\text{MAT}} + \Omega_{\text{VAC}} = 1 \quad \Omega_0 \equiv 1 \]

\[ H^2 = \frac{8\pi G}{3} (\rho_{\text{MAT}} + \rho_{\text{VAC}}) = H_0^2 \left[ \frac{\Omega_{\text{MAT}}}{a^3} + (1-\Omega_{\text{MAT}}) \right] \]

\[ \Rightarrow \quad a^2 = H_0^2 \left[ \frac{\Omega_{\text{MAT}}}{a^3} + (1-\Omega_{\text{MAT}}) \right] \]

\[ \Rightarrow \quad \frac{da}{dt} = H_0 \sqrt{\frac{3}{a}} \quad \text{or} \quad \int_0^t \frac{da}{\sqrt{a}} = H_0 \int_0^t dt = H_0 t_0 \]

\[ \Rightarrow \quad H_0 t_0 = \int_0^1 \frac{da}{\sqrt{\frac{3}{a^3} + (1-\Omega_{\text{MAT}}) a^2}} = \frac{2}{3} \frac{1}{\sqrt{1-\Omega_{\text{MAT}}}} \ln \frac{1+\sqrt{1-\Omega_{\text{MAT}}}}{\sqrt{1-\Omega_{\text{MAT}}}} \]

we see again that for matter dominated \( \Omega_{\text{MAT}} \to 1 \Rightarrow H_0 t_0 = \frac{2}{3} \ln(1+1/3^E) \]

but if \( \Omega_{\text{MAT}} \sim 0.25 \Rightarrow H_0 t_0 \approx 1.01 \Rightarrow t_0 \approx 6H_0 \approx 13.6 \text{ Byr} \)

which gives enough time for oldest stars to form.

Acceleration increases age as can be seen qualitatively:
Friedman eqn with curvature

So far we discussed the Friedman eqn without spatial curvature (i.e. \( k=0 \)). In the presence of curvature it reads,

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{k}{a^2}
\]

This equation can be thought of as the Newtonian analog of conservation of energy for a test particle on \( m \) on an expanding spherical region of mass \( M \) (with constant density \( \rho \)) and radius \( r \):

\[
E = \frac{1}{2} m v^2 - \frac{G M m}{r} = \frac{1}{2} m \dot{r}^2 - \frac{G M m}{r} = \text{const.}
\]

\[
\Rightarrow \quad \left( \frac{\dot{r}}{r} \right)^2 = \frac{2G M}{r^3} = \text{const.} \quad \text{or} \quad \left( \frac{\dot{\rho}}{\rho r} \right)^2 = \frac{8\pi G}{3} \rho + \text{const.}
\]

Thus the energy constant can be identified with the spatial curvature \( \kappa \).

We can rewrite the Friedman eqn as

\[
\frac{k}{a^2 H^2} = \frac{8\pi G}{3} \rho - 1
\]

which makes clear that the RHS defines a characteristic density \( \rho_{\text{crit}} \) for which the universe is spatially flat (\( k=0 \)), i.e. the critical density

\[
\rho_{\text{crit}} = \frac{3H^2}{8\pi G} \Rightarrow \frac{k}{a^2 H^2} = \frac{\rho}{\rho_{\text{crit}}} - 1
\]

It's customary to define the total \( \Omega \):

\[
\Omega \equiv \frac{\rho}{\rho_{\text{crit}}} \quad \Rightarrow \quad \frac{k}{a^2 H^2} = \Omega - 1 \quad \text{or} \quad \Omega = \frac{\rho}{\rho_{\text{crit}}} - \frac{k}{a^2 H^2}
\]

clearly \( \Omega = \frac{\Omega - \Omega_i}{1 - \Omega_i} \) and we have

\[
\frac{\Omega_i}{\Omega} = 1 + \frac{k}{a^2 H^2}
\]

defining \( \Omega_{\text{today}} = \frac{k}{a^2 H^2} \) today curvature contribution to \( \Omega \) today.
we obtain (today, i.e. $t=0$)

$$\sum \Omega_i + \Omega_{\text{vac}} = 1$$

so the sum of all generalized forms of $\Omega$'s add up to 1 (as before for flat case). Observationally, we know $\Omega$ is very close to 1 (explain briefly why).

Going back to the definition of the effective potential, we see that now when $k \to 0$ we have

$$\frac{\dot{a}^2}{H_0^2} + V_{\text{eff}}(a) = \frac{\Omega_k}{2}$$

with same definition as before:

$$V_{\text{eff}} = -\frac{1}{2} \left( \frac{\Omega_{\text{vac}} a^2}{a^2} + \frac{\Omega_{\text{m}}}{a} + \frac{\Omega_{\text{r}}}{a^2} + \Omega_{\text{v}} \right)$$

Qualitatively, one can explore the evolution of $\Omega_k$ for different cosmological parameters $H_0$, $\Omega_{\text{m}}$, $\Omega_{\text{r}}$, $\Omega_{\text{v}}$, $\Omega_{\text{vac}}$ by looking at the shape of the effective potential,

$$k=0 \Rightarrow \Omega_c = 0$$

since "Energy" always lives in $V_{\text{eff}}$, universe starts from big bang, decelerates until it gets dominated by $\text{vac}$ and accelerates, expands forever.

In this case, starting from a big bang, we expand until turning point, then recollapse to a big crunch!