As we already discussed, the equality of the inertial and gravitational mass has profound implications for understanding gravity. First, it implies that trajectories in gravitational fields are universal, independent of characteristics of the object. This makes a geometric approach to gravity possible.

Second, $mg = m\ddot{a}$ implies that locally a gravitational field is equivalent to being in an accelerating frame,

\[
\begin{align*}
\text{on Earth} & \quad \downarrow \\
\text{on the Moon} & \quad \downarrow
\end{align*}
\]

So, by performing mechanical experiments with particles, the observer cannot really tell whether the laboratory is in an accelerated frame in empty space, or an unaccelerated frame in a uniform gravitational field. But what about other types of experiments, i.e., with massless particles (e.g., photons) or electromagnetic fields, or nuclear forces? Einstein postulated that no experiment can (locally) distinguish uniform acceleration & gravitational fields:

this is the equivalence principle (EP)

The power of the EP derives from its generality, that it applies to all laws of physics. It implies, for example, that light bends in a gravitational field, as this is what it does.
From the point of view of this observer, the trajectory of the light ray is bent, by EP, so does light in a gravitational field.

The EP also provides a connection between GR and SR. A (locally) inertial frame can be obtained everywhere at any event by just taking a freely falling frame: in such a frame any particle not subject to any force except gravity will stay at rest or move at constant velocity (apple of inertia), like the shuttle astronauts. Due to EP, these frames are locally equivalent to inertial frames where special relativity holds. Therefore, in a local inertial frame, the laws of physics take the same form as in SR.

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GRAVITATIONAL TIME DILATION AND GRAVITATIONAL REDSHIFT

The EP leads to important insights, in particular that gravitational fields affect clocks and that such effects...
cannot be incorporated into SR, leading to a modification of space-time geometry.

Consider the following example: a rocket is uniformly accelerating in empty space, and two observers (A & B) measure the rate of emission (A) and reception (B) of light signals:

We will assume for simplicity that over the time of interest \( \frac{V}{c} \ll 1 \) and \( \frac{gh}{c^2} \ll 1 \) (where \( h \) is the proper length of the rocket). We work to linear order in \( \frac{V}{c} \) and \( \frac{gh}{c^2} \), thus we can neglect relativistic time dilation and Lorentz contraction (that enter to order \( O(c^2) \)), and \( \frac{gh}{c^2} \) small means that the rocket does not accelerate to relativistic speed in the time it takes a photon to go from A to B. We can then use simple mechanics,

\[
\begin{align*}
Z_{B(t)} &= \frac{1}{2}gt^2 \\
Z_{A(t)} &= h + \frac{1}{2}gt^2
\end{align*}
\]

The first pulse satisfies \( c\Delta t_A = Z_A(0) - Z_B(t_f) \),
the second pulse satisfies \( c(t_f + \Delta t_B - \Delta t_A) = Z_A(\Delta t_A) - Z_B(t_f + \Delta t_B) \).
\[ C(t) = h - \frac{1}{2} g t^2 \]

\[ \left( h_B + \Delta h_B - \Delta h_A \right) = h + \frac{1}{2} g \left( \Delta t_A \right)^2 - \frac{1}{2} g \left( h_B + \Delta h_B \right)^2 \]

\[ \Delta h_B = \Delta h_A - g \frac{h}{c^2} \Delta t_B \]

\[ \Delta h_B \approx \Delta h_A \left( 1 - g \frac{h}{c^2} \right) \quad \text{(to first order in } \Delta t \text{'s)} \]

\[ \frac{1}{c^2} \approx \hbar \nu \left( 1 + \frac{\hbar}{c^2} \right) \]

so reception times are smaller at B

Since one can think of the crests of a light wave of frequency \( \nu \) as a series of signals emitted at rate \( \nu \times 1/\Delta t \) we have that

\[ \nu_B \approx \left( 1 + g \hbar/c^2 \right) \nu_A \]

Thus light received at B is bluesthifted compared to A, or (higher frequency)

A is redshifted compared to B.

Due to the equivalence principle, these effects are also present in a gravitational field, therefore we obtain that in a gravitational field there is gravitational time dilation and redshift. We can rewrite these expressions in a simpler form introducing that \( gh \) is the gravitational potential difference between A & B

\[ \Delta \nu = \nu_A - \nu_B = gh \]
\[ \Delta \tau_A = \left(1 + \frac{\Delta \phi}{c^2}\right) \Delta \tau_B \]

\[ \gamma_B = \left(1 + \frac{\Delta \phi}{c^2}\right) \gamma_A \]

Thus, photons get redshifted as they climb in a gravitational field ($\gamma_A < \gamma_B$).

This result can also be expected from conservation of energy. Consider a simple thought experiment: a particle decays into 2 photons at low altitude, the photons recombine to the particle at higher altitude (thanks to mirrors):

\[
\begin{array}{c}
\text{mirror} \\
\hline
\\
\text{mirror} \\
\hline
\end{array}
\]

\[
\begin{array}{c}
\text{mirror} \\
\hline
\\
\text{mirror} \\
\hline
\end{array}
\]

\[
\begin{array}{c}
\text{mirror} \\
\hline
\\
\text{mirror} \\
\hline
\end{array}
\]

The initial energy is: \( E_0 = mc^2 = 2h\nu \)

Final energy is: \( E_f = mc^2 + mgH = 2h\nu + mgH > E_0 \)

To satisfy energy conservation, we must have that photons get redshifted as they climb the gravitational field, indeed:

\[ 2h\nu_0 = 2h\nu_f + mgH \]

\[ \text{where } mc^2 = 2h\nu_f \]

\[ \Rightarrow \gamma_0 = \gamma_f + \frac{gH}{c^2} \nu_f \]

\[ \Rightarrow \frac{\nu_f}{\nu_0} = \left(1 - gH/c^2\right) \]