We saw last lecture that we are forced by the equivalence principle (and the gravitational redshift) to consider general metric $g_{\mu\nu}$ instead of Minkowski $g_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. The line element is then

$$\text{d}s^2 = g_{\mu\nu} \text{d}x^\mu \text{d}x^\nu$$

where the metric tensor $g_{\mu\nu} = g_{\mu\nu}(x^\alpha)$ is a function of spacetime coordinates $x^\alpha$.

In GR coordinates $x^\alpha$ are arbitrary, and we can write the laws of physics in any coordinate system we want (not just inertial frames connected to each other by Lorentz transformations).

There is, of course, special frames (known as locally inertial frames) which correspond to free-falling where the laws of physics take the same form as special relativity (locally). However, as we shall see, this is only valid at a particular point in spacetime, in general there will be no class of coordinate system which simplifies unifying the laws of physics at all points in spacetime. Thus, we are interested in formulating laws in arbitrary coordinate systems.

Now, a line element specifies a geometry, but many different line elements may describe the same spacetime geometry; this is because we can always change coordinates and physics (geometry) should not change.

The simplest example of this is flat spacetime, which in Cartesian coordinates reads,
but in spherical coordinates is
\[ ds^2 = -dt^2 + dr^2 + r^2 \, d\theta^2 + r^2 \sin^2 \theta \, d\phi^2 \]
which gives
\[ g_{\alpha \beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{pmatrix} \]
where \( x^\alpha = (t, r, \theta, \phi) \)

Both line elements (though different) describe exactly the same geometry. In this case one goes from one to the other by just a spatial transformation, i.e.,
\[
\begin{cases}
x = r \cos \theta \cos \phi \\
y = r \sin \theta \cos \phi \\
z = r \sin \theta
\end{cases}
\]
but in more complicated change of coordinates are allowed (e.g., mixing space with time), which describe of course the same geometry in different variables.

Therefore, although \( g_{\alpha \beta} \) has 16 components (since it is a symmetric 4x4 matrix), there are only 6 independent physical components:

\[ g_{\mu \nu} : \text{10 components} - 4 \text{ coordinate transformations} = 6 \text{ indip. components that are physical} \]

Now, the EP says that locally any curved spacetime is equivalent to flat space, i.e., we can choose a local inertial frame (LIF) where special relativity holds locally. What we mean specifically is that we can perform a coordinate transformation (equivalent to jumping into free fall) such that
\[ x^\alpha \rightarrow x'^\alpha, \quad g_{\mu \nu} \rightarrow g'_{\mu \nu} = \eta_{\mu \nu} \]
at some point $P$ in spacetime (that's what is meant by local).

So far this may look trivial, it is always possible to diagonalize a symmetric real matrix, the statement above only adds that in that basis the diagonal elements (after trivial rescalings of coordinates) can only be $-1$ (for time) and $+1$ for space, i.e. there are one time-like and three spatial dimensions.

What happens in the neighborhood of $P$? It's easy to see that one can choose the $x^\alpha$ so that also first derivatives vanish, i.e.

$$\left( g^{\mu\nu} \right)_\rho = \eta^{\mu\nu} \quad \text{and} \quad \frac{\partial g^{\mu\nu}}{\partial x^\sigma} \bigg|_\rho = 0$$

That's the definition of a LIF.

You can see that then in a neighborhood of $P$, deviations of the metric from Minkowski only appear at second order in displacement $x^\alpha - x^\alpha(P)$, which is a small quantity. That's precisely what we mean by that locally we are in an FF.

If it were possible to choose coordinates such that globally $g^{\mu\nu} = \eta^{\mu\nu}$ then spacetime would be flat!

To see why the statement above applies only up to first derivatives, consider how the metric changes under a coordinate transformation $x'^\alpha = x'^\alpha(x^\beta)$, where we need to specify the four functions $x'^\alpha(x^\beta)$ at the old coordinates $x^\beta$.
\[ ds^2 = g_{\mu \nu} \, dx^\mu \, dx^\nu = g^{\alpha \beta} \, dx^\alpha \, dx^\beta \]

to obtain \( g^{\alpha \beta} \) in terms of \( g_{\mu \nu} \) we use the chain rule to write \( dx^\mu \) in terms of \( dx^\alpha \):

\[ dx^\mu = \frac{\partial x^\mu}{\partial x^\alpha} \, dx^\alpha \]

(cnote summation convention over \( \alpha \), superscript index in denominator acts as subscript, repeated indices appear only once "up" and "down")

Then we have,

\[ g_{\mu \nu} \frac{\partial x^\mu}{\partial x^\alpha} \frac{\partial x^\nu}{\partial x^\beta} = g^{\alpha \beta} \, dx^\alpha \, dx^\beta \]

\( \Rightarrow \) since this holds for arbitrary \( dx^\alpha \) displacements

\[ g^{\alpha \beta} = g_{\mu \nu} \frac{\partial x^\mu}{\partial x^\alpha} \frac{\partial x^\nu}{\partial x^\beta} \]

Now consider a coordinate transformation in the vicinity of \( P \),

\[ x^\mu (x') = x^\mu (x'^\alpha) + \frac{\partial x^\mu}{\partial x^\alpha} \bigg|_P (x'^\alpha - x'^\alpha)_P + \frac{1}{2} \frac{\partial^2 x^\mu}{\partial x'^\alpha \partial x'^\beta} \bigg|_P (x'^\alpha - x'^\alpha)_P (x'^\beta - x'^\beta)_P + \ldots \]

In the vicinity of \( P \), therefore, the transformation is specified by the constants \( \frac{\partial x^\mu}{\partial x^\alpha} \bigg|_P \), \( \frac{\partial^2 x^\mu}{\partial x'^\alpha \partial x'^\beta} \bigg|_P \), etc. - i)To obtain that

\[ g^{\mu \nu} = g_{\mu \nu} \]

we need 40 conditions, and from

\[ \frac{\partial x^\mu}{\partial x'^\alpha} \bigg|_P \]

we have 46 numbers we can choose,
So in fact we have an extra 6 degrees of freedom, which correspond to Lorentz boosts and spatial rotations that one the metric is in Minkowski form, it will remain so under these:

1) To get that \( \frac{\partial g_{\mu\nu}}{\partial x^\lambda} \bigg|_p = 0 \) we need to impose \( 10 \times 4 = 40 \) conditions, and from the next term \( \frac{\partial^2 g_{\mu\nu}}{\partial x^\lambda \partial x^\beta} \bigg|_p \) we have exactly 40 numbers at our disposal, so this is always possible!

2) However, if we want to make the second derivatives of the metric vanish, we need to impose

\[
\frac{\partial g_{\mu\nu}}{\partial x^\lambda \partial x^\delta} \bigg|_p = 0
\]

which correspond to \( 10 \times 10 = 100 \) conditions. On the other hand, we have from \( \frac{\partial^2 g_{\mu\nu}}{\partial x^\lambda \partial x^\beta} \bigg|_p \) only 80 numbers at our disposal (4 \( \times 20 \), the 20 are all symmetric possible combinations of \( \partial \), \( \partial \), \( \partial \), \( \partial \), try it!) – so this is in general not possible, we fall short by 20 “degrees of freedom” (which turn out to be the independent components of the Riemann curvature, as we shall see later). Thus, in general a LIF will have deviations from Minkowski that will depend on second derivatives of the metric, which will define spacetime curvature! (These will reduce, in the weak-field limit, to the old-fashioned tidal gravitational forces \( \nabla_i \nabla_j \phi \), where \( \phi \) is Newtonian potential.)
The interval $d\mathbf{s}^2$ can be calculated in any frame we want, since it is an invariant. In particular, in a LIF it takes a simple form which is the same (locally) as in SR.

Therefore, SR inherits the local light-cone structure of SR, i.e. we can divide a pair of space-time points separated by infinitesimal displacements $dx^a$ according to the sign of $d\mathbf{s}^2$:

\[ d\mathbf{s}^2 < 0 \text{ (time-like), } d\mathbf{s}^2 = 0 \text{ (light-like), } d\mathbf{s}^2 > 0 \text{ (space-like), or null} \]

Light moves along null curves with $d\mathbf{s}^2 = 0$, i.e. massive particles move along time-like curves that can be parametrized by proper time (now calculated using full metric):

\[ T_{AB} = \int_\alpha^\beta \left[ -g_{\alpha\beta} \, dx^\alpha \, dx^\beta \right]^{1/2} \]

where $T_{AB}$ is the integral along worldline.

Time-like curves always lie at any point inside the local lightcone, that guarantees particles are moving at speeds less than light at that point. Because spacetime is curved, lightcones will not make 45° angles with the x,t axes though. You can find what light does by solving $\times v = 0$ from $d\mathbf{s}^2 = 0$ once a metric is specified.

Now, given a metric we can compute not only lengths but also areas, 3-volumes and 4-volumes. The length is the interval, for areas let us consider the special case of diagonal metrics, which is all we'll need in this course:

\[ d\mathbf{s}^2 = g_{00} (dx^0)^2 + g_{11} (dx^1)^2 + g_{22} (dx^2)^2 + g_{33} (dx^3)^2 \]
In this case things are easier to see since coordinates are orthogonal. Consider for example the calculation of an area corresponding to a surface defined by \( x^0 = \text{const} \) and \( x^3 = \text{const} \). (i.e. \( t = \text{const} \), \( z = \text{const} \)) Since the metric converts from coordinate variations to lengths, the area in this \( x^1 - x^2 \) (\( x-y \)) plane is simply

\[
\frac{\text{d}A_{03}}{\text{d}x^0} = \text{proper length in } x^1 \times \text{proper length in } x^2 \\
= \sqrt{g_{11}} \, dx^1 \times \sqrt{g_{22}} \, dx^2
\]

and similarly for 3-volume (specified by \( x^0 = \text{const} \))

\[
\frac{\text{dV}_{01}}{\text{d}x^0} = \sqrt{g_{00} g_{11} g_{22} g_{33}} \, dx^1 dx^2 dx^3
\]

while for 4-volume one has to be slightly more careful

\[
\frac{\text{dV}}{\text{d}x^0} = \sqrt{-g_{00} g_{11} g_{22} g_{33}} \, dx^0 dx^1 dx^2 dx^3
\]

since otherwise the sqrt will give imaginary values (remember in a LTF \( g_{00} = -1 \)) so we need to add a minus sign to make volume real. In fact in this basis where metric is diagonal we see that \( \frac{\text{dV}}{\text{d}x^0} = \sqrt{-g} \, dx \), where \( g \) is the determinant of \( g_{\mu\nu} \). That's included the general expression when the metric is no longer diagonal (and similar for 3-volumes and areas where one uses the determinant of the "induced" metric on the 3-vol or surface).

You, in fact, have already seen this when calculating...
...volumes in spherical coordinates! Indeed, in flat space with spherical coordinates, we have
\[ ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \]

\[ \Rightarrow dA = \frac{\sqrt{g_{\theta\theta} g_{\phi\phi}}}{\sin \theta} \ d\theta \ d\phi = \frac{r^2}{\sin \theta} \ d\theta \ d\phi = r^2 \sin \theta \ d\theta \ d\phi \]

and for 3-volume:
\[ dV = \frac{\sqrt{|g|}}{\sin \theta} \ dr \ d\theta \ d\phi \]
\[ = r^2 \sin \theta \ dr \ d\theta \ d\phi \]

which should look very familiar!

Consider a slightly more interesting example, given by a homogeneous and isotropic closed universe, which in spatial metric reads
\[ ds^2 = \frac{dr^2}{1 - \frac{r^2}{a^2}} + r^2 (d\theta^2 + \sin^2 \theta \ d\phi^2) \]

**Circumference around the equator** (\(r=R; \ \theta = \pi/2\))
\[ C = \int ds = \int_0^{2\pi} r \ d\phi = 2\pi R \]

**Distance from \( r=R \) to \( r=2R \) along \( \theta = \text{const.} \) \( \phi = \text{const.} \)**
\[ s = \int ds = \int_0^R \frac{dr}{\sqrt{1 - \frac{r^2}{a^2}}} = a \arcsin \left( \frac{R}{a} \right) \]

**Area of surface** \( r=R \)
\[ A = \int dA = \int_0^{2\pi} d\theta \int_0^{\pi/2} r^2 \sin \theta \ d\theta \ = 4\pi R^2 \]

**Volume inside** \( r=R \)
\[ V = \int dr \int_0^{2\pi} d\theta \int_0^{\pi/2} r^2 \sin \theta \ \frac{r^2 \sin \theta}{\sqrt{1 - \frac{r^2}{a^2}}} \]
\[ = 4\pi a^3 \left[ \frac{1}{2} \arcsin \left( \frac{R}{a} \right) - \frac{R^2}{2a^2} \sqrt{1 - \left( \frac{R}{a} \right)^2} \right] \]
Vectors in curved spacetimes

To define vectors (and by this I mean 4-vectors, as usual) in curved spacetime is a bit tricky, as one has to be careful to recognize that vectors can only be defined locally, as measured by observers in a small patch of spacetime.

A useful example is given by the wind velocity at ground level on the surface of the Earth, due to Earth's curvature at each point on its surface vectors live on a tangent space, where we can do the usual operations of additions, multiplications, etc.

Vector fields defined at different points however, belong to different tangent spaces and thus there is no way of adding vectors at different points as we do in flat spacetime (where all tangent spaces coincide). So we have to abandon, for example, the notion of a displacement vector except for infinitesimally separated points where a single tangent space suffices.

As usual we write vectors in a basis

\[ \vec{a} = a^x \hat{e}_x \]

where we explicitly write down this is valid at spacetime point \( x^a \). The scalar product between vectors reduces as usual to the scalar product between basis vectors (up to components), i.e.

\[ \vec{a} \cdot \vec{b} = a^x b_\beta \hat{e}_x \cdot \hat{e}_\beta \]
There are two choices of basis vectors that are commonly used, namely:

i) \( \vec{e}_\alpha \cdot \vec{e}_\beta = \eta_{\alpha \beta} \) \hspace{1cm} (orthonormal basis)

ii) \( \vec{e}_\alpha \cdot \vec{e}_\beta = \gamma_{\alpha \beta} \) \hspace{1cm} (coordinate basis)

For orthonormal basis, we use indices with hats to stress that we are in a basis where all basis vectors are orthogonal and unit size, note the use of \( \eta_{\alpha \beta} \) is needed as the first basis vector must be timelike.

The most common example of an orthonormal basis is actually the observer's lab, in which one can think of its 4-velocity defining the timelike unit vector \( \vec{e}_0 = \vec{u}_{\text{obs}} \) and \( \vec{e}_\alpha \) are the three unit vectors in the spatial direction. This is important because components in this coordinate basis correspond to measurable quantities, e.g., we can write 4-momentum \( \vec{p} = p^\alpha \vec{e}_\alpha \) where the observed energy \( (p^0 = \mathcal{E}) \) can be obtained by

\[
\mathcal{E} = -\vec{p} \cdot \vec{e}_0 = -\vec{p} \cdot \vec{u}_{\text{obs}}
\]

In a coordinate basis, the basis vectors are not necessarily unit in size, nor orthogonal. (Note however, that the difference between orthonormal coordinate basis disappears in flat space.)

A familiar example is the condition for normalization of four velocity:

\[
-t = \vec{u} \cdot \vec{u} = g_{\alpha \beta} u^\alpha u^\beta = g_{00} \frac{dx^0}{d\tau} \frac{dx^0}{d\tau} = -\frac{ds^2}{dt^2}
\]
So you can see that when we write that

the components of \( \ddot{u} \) are \( \frac{dx^a}{dt} \) we are implicitly

assuming a coordinate basis, i.e.

\[
\ddot{u} = \frac{dx^a}{dt} \mathbf{e}_a
\]

with \( \mathbf{e}_a \cdot \mathbf{e}_b = g_{ab} \)

so that \( \ddot{u} - \ddot{u} = g_{ab} \frac{dx^a}{dt} \frac{dx^b}{dt} = -1 \)

Typically, calculations will be often easier in coordinate basis, and measurements are related to orthonormal basis.

A simple example is given by polar coordinates in 2D, where

\[
ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\phi^2 \quad \Rightarrow \quad \begin{cases} g_{rr} = 1 \\ g_{\phi\phi} = r^2 \end{cases}
\]

orthonormal basis has \( |\mathbf{e}_r| = |\mathbf{e}_\phi| = 1 \)

coordinate basis has \( |\mathbf{e}_r| = 1 \) but \( |\mathbf{e}_\phi| = r \) (\( = |\mathbf{e}_\phi| \))

so that \( \mathbf{e}_r \cdot \mathbf{e}_\phi = g_{r\phi} \)

\[\text{Hyper-surface}\]

A 3-dm hyper-surface or 3-surface can be defined in spacetime by a constraint of the form

\[ f(x^a) = 0 \]

e.g. This may be simplified in some cases as the clear
condition \( x^0 = h(x^1, x^2, x^3) \) which says given \( x^1, x^2, x^3 \)
at what \( x^0 \) is the "surface"
At each point on the 3-surface we can define normal (\( \hat{n} \)) and tangent (\( \tilde{e} \)) vectors so that

\[
\hat{n} \cdot \tilde{e} = 0
\]

Space-like surfaces have time-like normals \( \hat{n} \) and space-like tangent vectors \( \tilde{e} \), i.e., \( \hat{n} \cdot \tilde{e} < 0 \) (space-like surface).

There are many ways of defining space-like surfaces, of course, and these can "foliate" space-time into space & time, e.g.

Null surfaces are important as they are generated by light rays. Because light travels along null worldlines, the tangent vector along the light ray \( \tilde{e} \) satisfies

\[
\tilde{e} \cdot \tilde{e} = 0
\]

and there are 2 more tangent vectors which are orthogonal to \( \tilde{e} \) and space-like. Note \( \tilde{e} \) is also normal to the surface, because \( \tilde{e} \cdot \tilde{e} = 0 \) for all other tangent vectors and also \( \tilde{e} \cdot \tilde{e} = 0 \) where the second \( \tilde{e} \) here is the \( \tilde{e} \) tangent along the light ray. So null surfaces are a limit of space-like when \( \hat{n} \cdot \tilde{e} \to 0 \Rightarrow \hat{n} \cdot \hat{n} = 0 \)
A 2D example illustrates this point.

Note that null surfaces have a "one-way property". Once a time-like curve crosses it, then it cannot cross again. This will become important when we discuss Black Holes.