Exercise 1: Change of bases
Given two bases $e_{(\mu)}$ and $e_{(\mu')}$, related through $e_{(\mu')} = \Lambda^\mu_{\mu'} e_{(\mu)}$, show that their dual bases $e^{(\mu)}$ and $e^{(\mu')}$ are related through $e^{(\mu)} = \Lambda^\mu_{\mu'} e^{(\mu')}$. [If the notation displeases you, feel free to use a better one]. Explicitly derive the rules of transformations of the components of a tensor of rank $(1, 1)$.

Exercise 2: Orthonormal bases
Given a vector space of dimension $n$ and a metric tensor on that space [i.e. a symmetric, non-degenerate tensor of rank $(0, 2)$], prove that there exist bases in which $g_{\mu\nu} = \pm \delta_{\mu\nu}$.

Hint: work with the quadratic form $Q(X^1, \ldots, X^n) \equiv g_{\mu\nu} X^\mu X^\nu$ and proceed recursively. First consider the case where one of the diagonal components does not vanish, say $g_{11}$, and find a new coordinate $\bar{X}^1$ such that $Q(X^1, \ldots, X^n) = (\bar{X}^1)^2 + \bar{Q}(X^2, \ldots, X^n)$. Second, consider the case where all the diagonal components vanish, and suppose $g_{12} \neq 0$. Find new coordinates $\bar{X}^1, \bar{X}^2$ such that $Q(X^1, \ldots, X^n) = (\bar{X}^1)^2 - (\bar{X}^2)^2 + \bar{Q}(X^3, \ldots, X^n)$. Relate these new coordinates to new bases.

Exercise 3: Sylvester’s law of inertia
Suppose we have two orthonormal bases $e_{(\mu)}$ and $f_{(\nu)}$ of a $n$-dimensional vector space $\mathcal{V}$, in which the metric $g$ has signature [i.e. (number of -1's, number of +1's)] $(p, n-p)$ and $(q, n-q)$, respectively. Show that $p = q$.

Hint: Proceed by contradiction: if $p < q$, define the linear map $L : \mathcal{V} \rightarrow \mathbb{R}^{n-q+p}$ such that

$$L(V) \equiv (g(V, e_{(1)}), \ldots, g(V, e_{(p)}), g(V, f_{(q+1)}), \ldots, g(V, f_{(n)})).$$

Argue that there must exist a non-zero vector $V_0$ for which $L(V_0) = 0$. Compute the norm of this vector in two ways and show that it is positive and negative, hence zero. Then show that $V_0 = 0$, leading to a contradiction.

Exercise 4: The tangent space as the space of directional derivative operators
We denote by $\mathcal{F}$ the set of infinitely differentiable functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$. We define the tangent space at a point $P \in \mathbb{R}^n$ as the space of linear operators $V : \mathcal{F} \rightarrow \mathbb{R}$ that satisfy Leibniz’s rule, $V(f g) = f(P)V(g) + g(P)V(f)$.

Show that (i) these operators acting on constant functions give zero (ii) any such operator can be written as a linear combination of partial derivative operators, $V = V^\mu \partial_\mu$.

Hint: given $f \in \mathcal{F}$, show that at any point $P$ with coordinates $x^\mu_P$, there exists $n$ differentiable functions $H_\mu(x)$ s.t.

$$f(x) = f(x_P) + (x^\mu - x^\mu_P) H_\mu(x).$$

(1)

Prove (ii) using (i) and Leibniz rule applied to this expression.

Exercise 5: Levi-Civita tensor in spherical coordinates
Compute the all-covariant and all-contravariant components of the Levi-Civita tensor in the spherical polar coordinate basis of $\mathbb{R}^3$.

Exercise 6: index manipulation fun
(i) If the tensor $T_{\alpha \beta}$ is symmetric, show that $T^\alpha_{\beta} = T^\beta_{\alpha}$.

(ii) Write explicitly the antisymmetric part $T_{[\alpha \beta \gamma]}$ of the rank $(0, 3)$ tensor $T_{\alpha \beta \gamma}$.

(iii) Given a rank $(0, 2)$ tensor $T_{\alpha \beta}$, what is the rank of the tensor $T_{\alpha \beta} T_{\gamma}^{\sigma} T^{\beta \gamma}$? How about $T_{\alpha \beta} T_{\gamma}^{\alpha} T^{\beta \gamma}$?

(iv) Suppose that $n = 4$ and that, in a given basis where $g_{\mu\nu} = \eta_{\mu\nu}$, the components of the tensor $T_{\alpha \beta}$ are given by

$$T_{\mu\nu} = \begin{pmatrix} 2 & 0 & -1 & 1 \\ -3 & 0 & 2 & 1 \\ -2 & 1 & 0 & -1 \\ -1 & 0 & -2 & 3 \end{pmatrix}.$$ 

(2)

Compute $T_{\alpha \beta} T_{\gamma}^{\alpha} T^{[\beta \gamma]}$. Hint: think a little bit before starting!!