Exercise 1: Change of bases [3 points]
Given two bases $e_\mu$ and $e_\nu$, related through $e_\nu = \Lambda^\nu_\mu e_\mu$, show that their dual bases $e^{\mu}$ and $e^{\nu}$ are related through $e^{\mu} = \Lambda^\mu_\nu e^{\nu}$. Explicitly derive the rules of transformations of the components of a tensor of rank $(1,1)$.

Let us write $e^{(\mu)} = \Gamma^\mu_\nu e^{(\nu)}$, where \{\Gamma^\mu_\nu; \mu, \nu = 1, \ldots, n\} are the components of $e^{(\mu)}$ on the dual basis $e^{(\nu)}$. This implies

$$e^{(\mu)}(e^{(\nu)}) = \Gamma^\mu_\nu e^{(\nu)}(e^{(\nu)}) = \Gamma^\mu_\nu \delta^{\nu_'}_{\nu} = \Gamma^\mu_{\nu'}.$$  

(1)

On the other hand,

$$e^{(\mu)}(e^{(\nu)}) = e^{(\mu)}(\Lambda^\nu_\nu e^{(\nu)}) = \Lambda^\mu_\nu e^{(\mu)}(e^{(\nu)}) = \Lambda^\mu_\nu \delta^\nu_{\nu'} = \Lambda^\mu_{\nu'},$$

(2)

where we have used linearity of $e^{(\mu)}$ and the definition of the dual basis. Hence $\Gamma^\mu_{\nu'} = \Lambda^\mu_{\nu'}$ and $e^{(\mu)} = \Lambda^\mu_{\nu} e^{(\nu)}$.

The components of a $(1,1)$ tensor $T_{\alpha \beta}$, are defined as

$$T = T_{\mu \nu} e_\mu(\nu) \otimes e(\nu) = T_{\mu \nu} \Lambda^\mu_{\mu'} \Lambda^\nu_{\nu'} e^{(\mu)} \otimes e^{(\nu)} \equiv T_{\mu \nu} e^{(\mu)} \otimes e^{(\nu)},$$

(3)

implying

$$T_{\mu \nu} = \Lambda^\mu_{\mu'} \Lambda^\nu_{\nu'} T_{\mu' \nu'}.$$  

(4)

Common mistake: the metric is not involved at all in this proof! Several people wrote something like $e_\mu = g_{\mu \nu} e^{\nu}$. The meaning of the latter expression is as follows: let us write $\mathbf{e}$ the vector of components $e^\mu$ on some basis, and $\mathbf{e}$ the dual vector of components $e_\mu = g_{\mu \nu} e^{\nu}$ on the dual basis. In index-free notation, $\mathbf{e} = g(\mathbf{e}, -)$, which means that for any vector $V$, $\mathbf{e}(V) \equiv g(\mathbf{e}, V)$. Let us in particular apply this definition to a basis vector $e^{(\mu)}$, where $\mu$ is a label, NOT an index: we define $\mathbf{e}^{(\mu)}(V) \equiv g(e^{(\mu)}, V)$. In particular, $\mathbf{e}^{(\mu)}(e^{(\nu)}) = g(e^{(\mu)}, e^{(\nu)}) = g_{\mu \nu}$. This is NOT $\delta_{\mu \nu}$ in general. Keep parentheses on basis vectors, they have a meaning!

Exercise 2: Orthonormal bases [5 points]
Given a vector space of dimension $n$ and a metric tensor on that space \[i.e. a symmetric, non-degenerate tensor of rank (0,2),\] prove that there exist bases in which $g_{\mu \nu} = \pm \delta_{\mu \nu}$.

We define $Q(X^1, \ldots, X^n) \equiv g_{\mu \nu} X^\mu X^\nu$. Let us first prove that there exists an \textit{invertible} linear transformation $\hat{X}^\mu = \Lambda^\mu_\mu X^\nu$ such that $Q = \sum^\mu (\hat{X}^\mu)^2$. We proceed by induction. In one dimension, the metric just has one coefficient, $g_{11}$, which must be non-zero since the metric is non-degenerate. We define $\hat{X}^1 \equiv X^1/\sqrt{|g_{11}|}$, and get $Q = \pm (\hat{X}^1)^2$. This transformation is clearly invertible. Now suppose the theorem is true for any quadratic form of dimension $\leq n - 1$. Let us first consider the case where one of the diagonal coefficients, say $g_{11}$, is non-zero. We complete the squares:

$$Q = g_{11}(X^1)^2 + 2 \sum^\mu_{\mu > 1} g_{\mu 1} X^\mu X^1 + \sum^\mu_{\mu, \nu > 1} g_{\mu \nu} X^\mu X^\nu$$

$$= g_{11} \left( (X^1)^2 + 2 \sum^\mu_{\mu > 1} \frac{g_{\mu 1}}{g_{11}} X^\mu X^1 \right) + \sum^\mu_{\mu, \nu > 1} g_{\mu \nu} X^\mu X^\nu$$

$$= g_{11} \left( X^1 + \sum^\mu_{\mu > 1} \frac{g_{\mu 1}}{g_{11}} X^\mu \right)^2 + Q'(X^2, \ldots, X^n),$$

(5)

where $Q'$ is a quadratic form depending only on $X^2, \ldots, X^n$. By hypothesis, this form can be written in a normal form (i.e. with just plus and minus ones) in terms of some variables $\hat{X}^2, \ldots, \hat{X}^n$. We further define

$$\hat{X}^1 \equiv \frac{1}{\sqrt{|g_{11}|}} \left( X^1 + \sum^\mu_{\mu > 1} \frac{g_{\mu 1}}{g_{11}} X^\mu \right).$$

(6)
We see that \( Q \equiv \sum_{\mu} \pm (\tilde{X}^\mu)^2 \) in these variables. Moreover, the transformation is invertible: since the \((X^2, ..., X^n) \rightarrow (\tilde{X}^2, ..., \tilde{X}^n)\) transformation is invertible (by hypothesis of induction), all we have to show is that we can uniquely define \( X^1 \) in terms of \((\tilde{X}^1, ..., \tilde{X}^n)\), which is trivial.

Let us consider the case where all diagonal elements \( g_{\mu\nu} = 0 \) [note, this is not summed over!]. Since \( g \) is not the zero tensor, it has a non-zero off-diagonal component, say \( g_{12} \). So we have

\[
Q = 2g_{12} \left[ X^1 X^2 + \sum_{\mu > p} \frac{g_{\mu 1}}{g_{12}} X^\mu X^1 + \sum_{\mu > 1} \frac{g_{\mu 2}}{g_{12}} X^\mu X^2 \right] + \sum_{\mu > \nu > 2} g_{\mu\nu} X^\mu X^\nu \\
= 2g_{12} Y^1 Y^2 + Q'(X^3, ..., X^n), \quad Y^1 \equiv X^1 + \sum_{\mu > 2} \frac{g_{\mu 2}}{g_{12}} X^\mu, \quad Y^2 \equiv X^2 + \sum_{\mu > 2} \frac{g_{\mu 1}}{g_{12}} X^\mu \\
= \frac{1}{2} g_{12} \left[ (Y^1 + Y^2)^2 - (Y^1 - Y^2)^2 \right] + Q'(X^3, ..., X^n)
\]

We see that if we define \( \tilde{X}^1 \equiv \sqrt{2/[g_{12}]}(Y^1 + Y^2) \) and \( \tilde{X}^2 \equiv \sqrt{2/[g_{12}]}(Y^1 - Y^2) \), and use the fact that \( Q \) can be put in a normal form in the variables \((X^3, ..., \tilde{X}^n)\), we have shown once again that \( Q \equiv \sum_{\mu} \pm (\tilde{X}^\mu)^2 \), where \( \tilde{X}^\mu = \Lambda^\mu_\nu X^\nu \).

Here again, it is easy to show that this transformation is invertible (and explicitly find the inverse), given the fact that \((X^3, ..., \tilde{X}^n) \rightarrow (\tilde{X}^3, ..., \tilde{X}^n)\) is invertible.

Let us now pick some arbitrary basis \( e_{(\mu)} \). For an arbitrary vector \( X \), we have \( g(X, X) = g_{\mu\nu} X^\mu X^\nu \). We showed that we could find other coordinates \( \tilde{X}^\mu = \Lambda^\mu_\nu X^\nu \) such as this is a sum of \pm squares. We denote by \( \Lambda^\mu_\nu \), the inverse of the transformation \( \Lambda^\mu_\nu \). What we then have found is that

\[
g_{\mu\nu} \Lambda^\mu_\rho \Lambda^\rho_\nu \tilde{X}^\mu \tilde{X}^\nu = \sum_{\mu'} \pm (\tilde{X}^\mu)^2,
\]

for any \( \tilde{X}^\mu \). This implies \( \Lambda^\mu_\rho \Lambda^\rho_\nu g_{\mu\nu} = \pm \delta_{\mu\nu} \). We define the new basis \( e_{(\mu')} \equiv \Lambda^\mu_\nu e_{(\mu)} \). The components of \( g \) in this basis are \( \Lambda^\mu_\nu \Lambda^\nu_\rho g_{\mu\rho} = \pm \delta_{\mu\rho} \), which completes the proof.

**Exercise 3: Sylvester’s law of inertia [5 points]**

**Suppose we have two orthonormal bases \( e_{(\mu)} \) and \( f_{(\mu)} \) of a \( n \)-dimensional vector space \( V \), in which the metric \( g \) has signature \( [i.e. \ (\text{number of -1’s}, \ \text{number of +1’s})] \ (p, n-p) \) and \( (q, n-q) \), respectively. Show that \( p = q \).**

Let us order the \( e_{(\mu)} \) basis such that the first \( p \) diagonal components of the metric are \(-1\) and the last \( n-p \) components are \(+1\), and similarly for the \( f_{(\mu)} \).

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We proceed by contradiction. Let us suppose that \( p < q \), and define the linear map \( L : V \rightarrow \mathbb{R}^{n-q+p} \) such that

\[
L(V) = (g(V, e_{(1)}), ..., g(V, e_{(p)}), g(V, f_{(q+1)}), ..., g(V, f_{(n)})).
\]

Since \( p < q \), this linear map goes from a space of dimension \( n \) to a space of dimension \( < n \). If \( L(e_{(\mu)}) \neq 0 \) for all the basis vectors \( e_{(\mu)} \), the \( n \) vectors \( L(e_{(1)}), ..., L(e_{(n)}) \) must be linearly dependent, since \( \mathbb{R}^{n-q+p} \) has dimension \( < n \). This means that there exists \( n \) numbers \( \lambda^\mu \), not all zero, such that \( \lambda^\mu L(e_{(\mu)}) = 0 \). We define \( V_0 \equiv \lambda^\mu e_{(\mu)} \). This vector is non-zero and is such that \( L(V_0) = 0 \). Note that if \( L(e_{(\mu)}) = 0 \) for one of the \( e_{(\mu)} \) we can simply define \( V_0 = e_{(\mu)} \).

Let us now write \( V_0 = \lambda^\mu e_{(\mu)} = \kappa^\mu f_{(\mu)} \) [i.e. \( \lambda^\mu \) are the components of \( V_0 \) in the \( e_{(\mu)} \) basis, \( \kappa^\mu \) its components in the \( f_{(\mu)} \) basis]. \( L(V_0) = 0 \) implies that \( \lambda^\mu = 0 \) for \( \mu \leq p \) and \( \kappa^\mu = 0 \) for \( \mu \geq q+1 \) [each component of \( L(V_0) \) must vanish]. Hence

\[
V_0 = \sum_{\mu > p} \lambda^\mu e_{(\mu)} = \sum_{\mu \leq p} \kappa^\mu f_{(\mu)}.
\]

Since \( e_{(\mu)} \) and \( f_{(\mu)} \) are orthonormal bases, the norm of \( V_0 \) is such that

\[
g(V_0, V_0) = \sum_{\mu > p} (\lambda^\mu)^2 g(e_{(\mu)}, e_{(\mu)}) = \sum_{\mu \leq p} (\kappa^\mu)^2 g(f_{(\mu)}, f_{(\mu)}). \quad (10)
\]

But for \( \mu > p \), \( g(e_{(\mu)}, e_{(\mu)}) = +1 \), and for \( \mu \leq p \), \( g(f_{(\mu)}, f_{(\mu)}) = -1 \). Hence we find that

\[
g(V_0, V_0) = \begin{cases} \sum_{\mu > p} (\lambda^\mu)^2 & \geq 0 \\ -\sum_{\mu \leq p} (\kappa^\mu)^2 & \leq 0. \end{cases} \quad (11)
\]
The only way for these two conditions to be simultaneously satisfied is that $g(V_0, V_0) = 0 = \sum_{\mu \geq q}(\lambda^\mu)^2 = \sum_{\mu \leq q}(\kappa^\mu)^2$, hence all $\lambda^\mu$ and $\kappa^\mu$ are zero, hence $V_0$ vanishes (Note: the fact that $g(V_0, V_0) = 0$ alone is not enough to prove that $V_0 = 0$). But this is in contradiction with the fact that $V_0 \neq 0$.

We conclude that the hypothesis $p < q$ is incorrect. The same reasoning goes for $q < p$. So we conclude that $p = q$.

**Exercise 4: The tangent space as the space of directional derivative operators [5 points]**

We denote by $\mathcal{F}$ the set of infinitely differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$. We define the tangent space at a point $P \in \mathbb{R}^n$ as the space of linear operators $V : \mathcal{F} \to \mathbb{R}$ that satisfy Leibniz’s rule, $V(fg) = f(P)V(g) + g(P)V(f)$. Show that (i) these operators acting on constant functions give zero (ii) any such operator can be written as a linear combination of partial derivative operators, $V = V^\mu \partial_\mu$.

(i) Take a constant function $f_0$, then $V(f_0^2) = 2f_0V(f_0)$ by Leibniz’s rule. But since $f_0$ is a constant, then $V(f_0^2) = V(f_0, f_0) = f_0V(f_0)$, by linearity. Hence $f_0V(f_0) = 0$, so either $f_0 = 0$ [and $V(0) = 0$ by linearity], or $V(f_0) = 0$.

(ii) Let be a point $P$ with coordinates $x_P$. Given $x$, let’s define the function $F(t) \equiv f(x_P^\mu + t(x^\mu - x_P^\mu))$. This function is differentiable and satisfies

$$F(1) = F(0) + \int_0^1 \frac{d}{dt}F(t) = F(0) + (x^\mu - x_P^\mu) \int_0^1 \frac{d}{dt}f(x_P^\mu + t(x^\mu - x_P^\mu)).$$

In other words we have shown that

$$f(x) = f(x_P) + (x^\mu - x_P^\mu)H_\mu(x),$$

where the functions $H_\mu(x)$ are explicitly defined above, and infinitely differentiable. They satisfy $H_\mu(x_P) = \partial_\mu f(x_P)$.

We now define the constant function $f_P(x) = f(x_P)$. We also define the functions $\lambda^\mu(x) \equiv x^\mu - x_P^\mu$, so we have $f(x) = f_P(x) + \lambda^\mu(x)H_\mu(x)$. Let’s apply a vector $V$ to this expression

$$V(f) = V(f_P + \lambda^\mu H_\mu) = V(f_P) + \lambda^\mu(x_P) V(H_\mu) + H_\mu(x_P) V(\lambda^\mu) = \lambda^\mu H_\mu f(x_P),$$

where we used linearity of $V$, the fact that it vanishes on constant functions, Leibniz’s rule, and the fact that $\lambda_\mu(x_P) = 0$. Defining $V^\mu \equiv V(\lambda^\mu)$, we conclude that $V = V^\mu \partial_\mu$. So we have shown that any tangent vector can be written as a linear combination of partial derivative operators. These operators are obviously linearly independent, and so they form a basis [a coordinate basis] of the tangent space, which is therefore of dimension $n$.

**Exercise 5: Levi-Civita tensor in spherical coordinates [3 points]**

Compute the all-covariant and all-contravariant components of the Levi-Civita tensor in the spherical polar coordinate basis of $\mathbb{R}^3$.

It is easiest to start in the orthonormal basis $e_r = \partial_r$, $e_\theta = \frac{1}{r} \partial_\theta$, $e_\phi = \frac{1}{r \sin \theta} \partial_\phi$. This basis is positively oriented with respect to the cartesian coordinate basis ($\partial_x, \partial_y, \partial_z$), so in this basis the components of the Levi-Civita tensor are

$$\epsilon_r^\theta \phi = \epsilon_r^\phi \theta = 1,$$

and all other components are obtained by the antisymmetry of $\epsilon$.

The covariant components of the Levi-Civita tensor in the coordinate basis are

$$\epsilon_{r\theta \phi} = \epsilon(\partial_r, \partial_\theta, \partial_\phi) = \epsilon(e_r, r e_\theta, r \sin \theta e_\phi) = r^2 \sin \theta.$$

All other components are obtained by the antisymmetry of $\epsilon$. The infinitesimal volume of a parallelepiped with edges $\Delta r, \Delta \theta, \Delta \phi$ is then $\Delta r \Delta \theta \Delta \phi \epsilon_{r\theta \phi} = r^2 \Delta r \Delta \theta \Delta \phi$.

The contravariant components are

$$\epsilon^{r \theta \phi} = g^{r r} g^{\theta \theta} g^{\phi \phi} \epsilon_{r \theta \phi} = \frac{1}{r^4 \sin^2 \theta} r^2 \sin \theta = \frac{1}{r^2 \sin \theta},$$

where we raised indices with the inverse metric, which is diagonal. Again, all other components are obtained by antisymmetry.
Exercise 6: index manipulation fun [3 points]

(i) If the tensor $T_{\alpha\beta}$ is symmetric, show that $T^\alpha_\beta = T^\alpha_\beta$.

\[ T^\alpha_\beta = g^{\alpha\lambda} T_{\lambda\beta} \quad \text{[by definition]} \]
\[ = g^{\alpha\lambda} T_{\beta\lambda} \quad \text{[} T_{\alpha\beta} \text{ is symmetric]} \]
\[ = T^\alpha_\beta \quad \text{[by definition].} \] (18)

(ii) Write explicitly the antisymmetric part $T_{[\alpha\beta\gamma]}$ of the rank (0, 3) tensor $T_{\alpha\beta\gamma}$.

The permutations of $[1, 2, 3]$ with signature $-1$ are $[2, 1, 3], [1, 3, 2], [3, 2, 1]$; those with signature $+1$ are the identity, $[2, 3, 1]$, and $[3, 1, 2]$. Hence we have

\[ T_{[\alpha\beta\gamma]} = \frac{1}{6} [T_{\alpha\beta\gamma} + T_{\beta\gamma\alpha} + T_{\gamma\alpha\beta} - T_{\beta\alpha\gamma} - T_{\alpha\gamma\beta} - T_{\gamma\beta\alpha}] . \] (19)

Note: the symmetric part is

\[ T_{(\alpha\beta\gamma)} = \frac{1}{6} [T_{\alpha\beta\gamma} + T_{\beta\gamma\alpha} + T_{\gamma\alpha\beta} + T_{\beta\alpha\gamma} + T_{\alpha\gamma\beta} + T_{\gamma\beta\alpha}] . \] (20)

We can define $S_{\alpha\beta\gamma} \equiv T_{(\alpha\beta\gamma)} + T_{[\alpha\beta\gamma]} \equiv \frac{1}{3} (T_{\alpha\beta\gamma} + T_{\beta\gamma\alpha} + T_{\gamma\alpha\beta})$. In terms of this tensor, we have

\[ T_{[\alpha\beta\gamma]} = \frac{1}{2} (S_{\alpha\beta\gamma} - S_{\beta\alpha\gamma}) = S_{(\alpha\beta)\gamma} , \] (21)

\[ T_{(\alpha\beta\gamma)} = \frac{1}{2} (S_{\alpha\beta\gamma} + S_{\beta\alpha\gamma}) = S_{(\alpha\beta)\gamma} . \] (22)

As a consequence, if $S_{\alpha\beta\gamma} = 0$, then $T_{[\alpha\beta\gamma]} = 0$ and $T_{(\alpha\beta\gamma)} = 0$. This is the case for the divergence of the electromagnetic tensor $\partial_\alpha F_{\beta\gamma}$, so both the symmetric and antisymmetric part of this tensor vanish.

(iii) Given a rank (0,2) tensor $T_{\alpha\beta}$, what is the rank of the tensor $T_{\alpha\beta} T^\gamma_\sigma T^{\sigma\gamma}$? Rank (1, 1).

How about $T_{\alpha\beta} T^\gamma_\sigma T^{\sigma\gamma}$? Rank (0, 0), i.e. a scalar [all indices are contracted].

(iv) Suppose that $n = 4$ and that, in a given basis where $g_{\mu\nu} = \eta_{\mu\nu}$, the components of the tensor $T_{\alpha\beta}$ are given by

\[ T_{\mu\nu} = \begin{pmatrix} 2 & 0 & -1 & 1 \\ 3 & 0 & 2 & 1 \\ -2 & 1 & 0 & -1 \\ -1 & 0 & -2 & 3 \end{pmatrix} . \] (23)

Compute $T_{\alpha(\beta} T^\gamma_\sigma T^{\sigma\gamma)}$.

There is no need to do any calculation here: the (0,2) tensor $T_{\alpha(\beta} T^\gamma_\sigma$ is symmetric, while the (2,0) tensor $T^{[\beta\gamma]}$ is antisymmetric. Hence

\[ T_{\alpha(\beta} T^\gamma_\sigma T^{[\beta\gamma]} = -T_{\alpha(\gamma} T^\sigma_\beta T^{[\gamma\beta]} = -T_{\alpha(\beta} T^\gamma_\sigma T^{[\beta\gamma]} , \] (24)

where in the last equality we relabelled dummy indices. So we get $T_{\alpha(\beta} T^\gamma_\sigma T^{[\beta\gamma]} = 0$. 