Consider the following weak-field metric outside a source with mass $M$ and angular momentum $\vec{J}$:

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{4}{r^2}(\hat{x} \times \vec{J}) \cdot d\vec{x}dt + \left(1 + \frac{2M}{r}\right)d\vec{x}^2 \equiv (\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu. \quad (1)$$

Consider a massive and non-relativistic point particle on a geodesic, i.e. on a nearly Keplerian orbit about the central mass. Suppose it carries a spin $S^\alpha$ that is a purely spatial vector in the rest-frame of the particle, i.e. $S^\alpha U_\alpha = 0$.

This spin is parallel-transported along the particle’s geodesic, i.e. $U^\beta \nabla_\beta S^\alpha = 0$.

**Preliminary remarks: order-of-magnitude considerations.**

In what follows we wil expand to linear order in small metric perturbations and particle velocities. Let us first estimate how they relate to each other. If the source has characteristic size $R$ and rotational velocity $V \ll 1$, then $J \sim MRV$ and, at distances $r \sim R$, we have

$$\frac{J}{r^2} \ll \frac{M}{r} \ll v \ll 1. \quad (2)$$

The circular velocity of a particular in a (quasi-) Keplerian orbit has magnitude $v \sim \sqrt{M/r} \gg M/r$. Therefore we have the following ordering of small quantities:

$$\frac{J}{r^2} \ll \frac{M}{r} \ll v \ll 1. \quad (3)$$

This does not tell us how $vM/r$ compares to $J/r^2$:

$$\frac{J}{r^2} \sim \frac{RV}{r} \sim \frac{V}{v}. \quad (4)$$

We don’t know a priori how the characteristic rotation velocity compares to the circular velocity, so we’ll keep both terms.

(i) **Compute $S^0$ as a function of $S^i$, to lowest order in the particle’s velocity $\vec{v} \equiv \vec{d}\vec{x}/dt$ and metric perturbations.**

The orthogonality condition is

$$0 = g_{\mu\nu}U^\mu S^\nu = -(1 - h_{00})U^0 S^0 + h_{0i}(U^0 S^i + U^i S^0) + (\delta_{ij} + h_{ij})U^i S^j. \quad (5)$$

Divide by $U^0$ to get

$$-(1 - h_{00})S^0 + h_{0i}(S^i + v^i S^0) + (\delta_{ij} + h_{ij})v^i S^j, \quad (6)$$

implying, to lowest-order in metric perturbations,

$$S^0 = h_{0i} S^i + \vec{v} \cdot \vec{S}. \quad (7)$$

(ii) **Write explicitly the parallel-transport equation for $S^i$ in terms of coordinate time $t$, and to linear order in the particle’s velocity. For now keep the expression general in terms of Christoffel symbols.**

$$U^\mu \nabla_\mu S^i = U^\mu \partial_\mu S^i + \Gamma^i_{\mu\nu} U^\mu S^\nu = \frac{dS^i}{dt} + \Gamma^i_{\mu\nu} \frac{dx^\mu}{dt} S^\nu = 0. \quad (8)$$

Dividing by $U^0 = dt/d\tau$, we get

$$\frac{dS^i}{dt} = -\Gamma^i_{00} S^0 - \Gamma^i_{0\nu} v^\nu S^0 = -\Gamma^i_{00} S^0 - \Gamma^i_{0j} v^j S^j + O(v^2). \quad (9)$$
Substituting the result of question (ii), and neglecting terms quadratic in the metric perturbation, we get

\[
\frac{dS^i}{dt} = -\Gamma^i_{ij}S^j - (\Gamma^i_{00}v^k + \Gamma^i_{jk}v^j) S^k. \tag{10}
\]

(iii) Compute the relevant Christoffel symbols to lowest order in metric perturbations, and simplify the parallel-transport equation for \( S^i \). Express your final result in terms of \( x^{(i)}v^j \) and \( x^{(i)}v^j \).

The relevant Christoffel symbols are, to lowest order in metric perturbations,

\[
\begin{align*}
\Gamma^i_{00} &= - \frac{1}{2} h_{00,i} = \frac{M}{r^2} \hat{x}_i, \\
\Gamma^0_{ij} &= h_{0[i,j]}, \\
\Gamma^i_{jk} &= \frac{1}{2} (h_{ik,j} + h_{ij,k} - h_{jk,i}) = \frac{M}{r^2} \left[ \delta_{jk} \hat{x}_i - \delta_{ik} \hat{x}_j - \delta_{ij} \hat{x}_k \right].
\end{align*}
\]

Like in class, we define \( B_k \equiv \epsilon_{kji}h_{0i,j} = \epsilon_{kji}h_{0[i,j]} \), implying \( h_{0[i,j]} = \frac{1}{2} \epsilon_{ikj}B_k \).

Now \( h_{0i,j} = 2\epsilon_{ilm}x_l J_m/r^3 \), so

\[
\begin{align*}
h_{0i,j} &= \frac{2}{r^3} \epsilon_{ilm} J_m (\delta_{lj} - 3\hat{x}_l \hat{x}_j),
\end{align*}
\]

from which we get

\[
\tilde{B} = \frac{2}{r^3} (\tilde{J} - 3(\hat{x} \cdot \tilde{J}) \hat{x}).
\]

The parallel-transport equation then becomes

\[
\begin{align*}
\frac{dS^i}{dt} &= -\frac{1}{2} (\tilde{B} \times \tilde{S}^j) - \frac{M}{r^2} \left( 2 x^i v^k + \delta_{jk} \hat{x}^i v^j - \delta_{ik} \hat{x}^j v^j - \delta_{ij} \hat{x}^k v^j \right) S^k \\
&= -\frac{1}{2} (\tilde{B} \times \tilde{S}^j) - \frac{M}{r^3} \left( 2 x^i v^k - x^k v^i - \delta_{ik} \hat{x}^j \hat{v} \right) S^k. \tag{16}
\end{align*}
\]

Expressing \( x^i v^k \) and \( x^k v^i \) in terms of their symmetric and antisymmetric parts, we get

\[
\begin{align*}
\frac{dS^i}{dt} &= -\frac{1}{2} (\tilde{B} \times \tilde{S}^j) - \frac{M}{r^3} \left( 3 x^{[i} v^{k]} + x^{(i} v^{k)} - \delta_{ik} \hat{x}^j \hat{v} \right) S^k \\
&= -\frac{1}{2} (\tilde{B} \times \tilde{S}^j) - \frac{M}{r^3} \left( 3 x^{[i} v^{k]} + x^{(i} v^{k)} - \delta_{ik} \hat{x}^j \hat{v} \right) S^k. \tag{17}
\end{align*}
\]

(iv) Re-express the antisymmetric pieces in terms of the orbital specific angular momentum \( \vec{\ell} = \vec{x} \times \vec{v} \). Recall that \( \frac{d\vec{x}}{dt} = -\frac{M}{r^2} \hat{x} \) for a Keplerian orbit; use this to express the symmetric pieces \( x^{(i} v^{j)} \) in terms of a time derivative.

Again, we write

\[
x^{[i} v^{k]} = \frac{1}{2} \epsilon_{ikj} \ell_j, \tag{18}
\]

and replace the symmetric parts, to get

\[
\frac{dS^i}{dt} = -\frac{1}{2} (\tilde{B} \times \tilde{S}^j) - \frac{3 M}{2 r^3} \epsilon_{ikj} \ell_j S^k + \left( \ell^{(i} v^{k)} - \delta_{ik} \hat{v} \cdot \hat{v} \right) S^k = -\frac{1}{2} (\tilde{B} \times \tilde{S}^j) - \frac{3 M}{2 r^3} (\ell \times \tilde{S}^j)^i + \frac{1}{2} \frac{d}{dt} (v^i v^k - \delta_{ik} v^2) S^k \tag{19}
\]

(v) Finally, average over one circular orbit, assuming \( S^i \) changes little in one orbit (which you will confirm a posteriori). Use the fact that the orbit is periodic to argue that the total derivative term cancels. You should get something of the form

\[
\frac{\langle d\vec{S} \rangle}{dt} = \tilde{\Omega} \times \vec{S}, \tag{20}
\]

where \( \tilde{\Omega} \) contains two terms: one linear in \( M \vec{\ell} \) and the other linear in \( \tilde{J} \). The former is the \textbf{geodetic precession} and the latter is \textbf{Lense-Thirring precession}, applied this time to a gyroscope rather than to the orbital angular
momentum as we did in class. Estimate the two precession rates for a point mass orbiting the Earth at a 650-km altitude on a circular polar orbit (as is the case for the Gravity Probe B (GPB) satellite), and express them in arcsecond per year.

For reference, GPB has measured the geodetic precession to a fractional accuracy of 0.003 and the Lense-Thirring gyroscopic precession to a fractional accuracy of 0.2. The LAGEOS satellites have measured the Lense-Thirring precession of orbits to a fractional accuracy of 0.05.

The orbit being periodic, the average of any full time derivative vanishes. So the last term drops and we are left with

\[
\left\langle \frac{d\vec{S}}{dt} \right\rangle = -\frac{1}{2} \vec{B} \times \vec{S} + \frac{3}{2} \frac{M}{r^3} \vec{\ell} \times \vec{S}.
\]  

(21)

The average of the gravitomagnetic field is

\[
\frac{1}{2} \langle \vec{B} \rangle = \frac{1}{r^3} \left( \vec{J} - 3((\hat{x} \cdot \vec{J})\hat{x}) \right) = \frac{1}{r^3} \left( \vec{J} - \frac{3}{2} (\vec{J} - (\hat{\ell} \cdot \vec{J})\hat{\ell}) \right) = \frac{1}{2} \frac{1}{r^3} \left( 3(\hat{\ell} \cdot \vec{J})\hat{\ell} - \vec{J} \right).
\]  

(22)

The rate of geodetic precession is

\[
\Omega_{\text{geodetic}} = \frac{3}{2} \frac{M}{r^3} \ell.
\]  

(23)

For a mass on a circular orbit of radius \( r \), \( v = \sqrt{M/r} \) and \( \ell = \sqrt{Mr} \), so

\[
\Omega_{\text{geodetic}} = \frac{3}{2} \frac{M^{3/2}}{r^{5/2}}
\]  

(24)

For an altitude of 650 km, we have \( r \approx 7000 \) km, giving a geodetic precession rate of 6.6" per year.

For a polar orbit \( \ell' \) is orthogonal to \( \vec{J} \), so the Lense-Thirring precession rate is \( \Omega_{LT} = J/(2r^3) \). Assuming the Earth is a constant-density solid body, \( J \approx \frac{2}{5} MR_\oplus^2 \Omega_\oplus \), where \( \Omega_\oplus \) is the rotation rate of the Earth and \( R_\oplus \) is the Earth radius. Therefore, for \( r \approx R_\oplus \), we get, for a polar orbit,

\[
\Omega_{LT} \approx \frac{1}{5} \frac{M}{r} \Omega_\oplus \approx 0.06" \text{ yr}^{-1}.
\]  

(25)

This is a rough approximation of the Earth’s angular momentum and the correct value is 0.04" yr\(^{-1}\).