• Energy radiated in GWs

\[ g_{\mu\nu} = g_{\mu\nu}^{(1)} + g_{\mu\nu}^{(2)} = 8\pi T_{\mu\nu} \]

Define \( T_{\mu\nu} \equiv -\frac{1}{8\pi} G_{\mu\nu}^{(2)} \Rightarrow G_{\mu\nu}^{(1)} = 8\pi (T_{\mu\nu} + \nu_{\mu\nu}) \), \( \nu_{\mu\nu} \sim \delta_{\mu\nu} \delta \Sigma, \rho. \Sigma \).

The energy of GWs cannot be localized: one can always find a local coordinate system in which \( h_{\mu\nu} = 0, \rho \delta h_{\mu\nu} = 0 \Rightarrow \nu_{\mu\nu} = 0 \) to second-order.

However, it is well-defined when averaging over several wavelengths: \( T_{\mu\nu}^{GW} \equiv \langle \nu_{\mu\nu} \rangle \).

This is similar to what happens with any wave: though it is clear that waves carry energy and momentum, they are not localized, but only defined across several \( \lambda \).

Upon averaging \( \langle \partial_\lambda h_{\mu\nu} \partial_\lambda h_{\mu\nu} \rangle \).

A tedious calculation gives

\[ T_{\mu\nu}^{GW} = \frac{1}{32\pi} \left\langle \partial_{\rho\sigma,\mu} h^{\rho\sigma}_{\mu\nu} - \frac{1}{2} \partial_{\mu} h_{\nu\lambda} - \partial_{\nu} h_{\mu\lambda} + \partial_{\rho\sigma} h_{\mu\lambda} - \partial_{\rho\sigma} h_{\nu\lambda} \right\rangle \]

It can be shown that this is gauge-invariant: if \( h_{\mu\nu} \sim \xi_{\mu\nu} \sim \xi \), then \( T_{\mu\nu}^{GW} \rightarrow T_{\mu\nu}^{GW} (1 + O(\xi)) \) under gauge transformations.

For details of the averaging procedure and proof of gauge invariance: Isaacson '68

Far from source, dominated by TT part:

\[ T_{\mu\nu}^{GW} = \frac{1}{32\pi} \left\langle h_{\rho\sigma,\mu} h^{\rho\sigma}_{\mu\nu} \right\rangle \]
The power radiated, integrate \( T_{10}^{GW} \) over sphere of radius \( r \):

\[
P = - r^2 \int \, dx^2 \, T_{10}^{GW} = - \frac{1}{4} \left[ \left\langle P_{km} P_{km} + P_{km} P_{km} - P_{ll} P_{mm} \right\rangle \right] \langle \ddot{\Phi}_{ll} \ddot{\Phi}_{mm} \rangle \text{ (over angles)}
\]

\[
\left\langle P_{km} P_{km} \right\rangle = \alpha \delta_{kl} \delta_{mn} + \beta \left( \delta_{km} \delta_{kn} + \delta_{km} \delta_{mn} \right) \quad \text{(only possible isotropic tension!)}
\]

\[
\left\langle P_{ll} P_{mm} \right\rangle = 4 = 9 \alpha + 6 \beta
\]

\[
\left\langle P_{ll} P_{ee} \right\rangle = 2 = 3 \alpha + 12 \beta
\]

\[
\Rightarrow \left\langle P_{km} P_{km} + P_{km} P_{km} - P_{ll} P_{mm} \right\rangle \text{ (over angles)} = \frac{2}{5} \left( \delta_{km} \delta_{kn} + \delta_{km} \delta_{mn} - \delta_{ll} \delta_{mm} \right) + \frac{2}{15} \delta_{ll} \delta_{mn}
\]

Since \( \ddot{\Phi}_{ll} = 0 \Rightarrow \)

\[
P = - \frac{1}{5} \left\langle \ddot{\Phi}_{ll} \ddot{\Phi}_{ll} \right\rangle \quad \text{Quadrupole formula.}
\]

This is the analog of \( P = - \frac{2}{3} \langle \ddot{d} \ddot{d} \rangle \) for electromagnetism. \( \ddot{d} \) = charge dipole.

There is no mass-dipole radiation: \( \ddot{d} = \ddot{p} = \text{constant.} \)
Application: orbital decay of a circular binary.

\[
\Phi_3 \equiv \Phi_0 = 16 \left( \frac{M_1 M_2}{M_1 + M_2} \right)^2 a^4 N^2 \left\{ \sin^2 \left( \frac{2 \pi}{n} \right) \left( \hat{u} \cdot \hat{v} - \hat{u} \cdot \hat{w} \right) - \cos^2 \left( \frac{2 \pi}{n} \right) \left( \hat{u} \cdot \hat{v} + \hat{u} \cdot \hat{w} \right) \right\}
\]

\[
= 32 \left( \frac{M_1 M_2}{M_1 + M_2} \right)^2 a^4 N^2 \left\{ \sin^2 \left( \frac{2 \pi}{n} \right) + \cos^2 \left( \frac{2 \pi}{n} \right) \right\} = 32 \left( \frac{M_1 M_2}{M_1 + M_2} \right)^2 a^4 N^2
\]

\[
\dot{P} = - \frac{3 \pi}{5} \frac{M_1 M_2 (M_1 + M_2)}{a^5}
\]

\[
E = \frac{M_1 + M_2 - \frac{M_1 M_2}{2a}}{\alpha^3}
\]

\[
\dot{E} = \frac{M_1 M_2}{2a^4} \dot{a}
\]

(assuming \( \dot{\hat{n}} = 0 = \dot{\hat{\omega}} \))

\[
\dot{a} = \frac{2 \alpha^4}{M_1 M_2} \quad \frac{d}{dt} \left( \frac{\alpha^4}{M_1 M_2} (M_1 + M_2) \right)
\]

\[
\frac{d}{dt} \left( \frac{\alpha^4}{M_1 M_2} (M_1 + M_2) \right)
\]

Frequency of GWs:

\[
f = \frac{2 \alpha}{2 \pi} \quad \dot{\omega} \quad a^{-3/2} \quad \Rightarrow \quad f = f_0 \left(1 - \frac{t - t_0}{T_{\text{merge}}} \right)^{3/8}
\]

GW 170817

LIGO-Livingston

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\dot{f} = \frac{3}{8} \frac{1}{T_{\text{merge}}} \frac{f_0}{\left(1 - \frac{t - t_0}{T_{\text{merge}}} \right)^{1/8}}
```

\[
\frac{d}{dt} \left( \frac{\alpha^4}{M_1 M_2} (M_1 + M_2) \right)
\]

The "Chirp mass" is a particularly well-measured combination of masses.

For GW 170817: \( \mathcal{M}_{\text{chirp}} = 1.188^{+0.004}_{-0.002} \, M_\odot \) (Historically, \( f \) \& \( f^{3/5} \) first observed in Hulse-Taylor pulsar)
Dealing with fully relativistic sources

So far we assumed that \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \), \( |h_{\mu\nu}| \ll 1 \) everywhere.

What if the sources of GWS are black holes or neutron stars?

⇒ We only require spacetime to be asymptotically flat, i.e., the existence of coordinates s.t. \( h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} \ll 1 \) far from the source.

Might be \( \gg 1 \) inside sources.

Define \( G^{(n)}_{\mu\nu} = -\frac{1}{2} \left( \Box h_{\mu\nu} - 2 \partial^\kappa (h_{\mu\nu} \partial_\kappa) + \eta_{\mu\nu} \partial^\kappa h_{\partial_\kappa} + \partial_{\kappa} h - \eta_{\mu\nu} \partial^\kappa h \right) \) (linearized Einstein tensor).

Define \( t_{\mu\nu} = -\frac{1}{8\pi} \left( G_{\mu\nu} - G^{(n)}_{\mu\nu} \right) \)

The E.F.E takes the exact form \( G^{(n)}_{\mu\nu} = 8\pi \left( T_{\mu\nu} + t_{\mu\nu} \right) \equiv 8\pi T_{\mu\nu}^{\text{eff}} \)

\( \nabla^\mu G^{(n)}_{\mu\nu} = 0 \Rightarrow \nabla^\mu T_{\mu\nu}^{\text{eff}} = 0 \) (but \( \nabla^\mu t_{\mu\nu} = 0 \)).

We may pick our favourite gauge to solve (if formally) for \( h_{\mu\nu} \) given \( T_{\mu\nu}^{\text{eff}} \).

Far from the source, the metric takes the form

\[
\text{ds}^2 = -\left(1 - \frac{2M}{r}\right)\text{d}t^2 + 4 \frac{\left(\hat{r} \times \hat{\mathbf{e}}\right)}{\hat{r}^2} \, \text{d}\hat{r} \, \text{d}t + \left(1 + \frac{2M}{r}\right) \, \text{d}\hat{r}^2 + \hat{k}_{ij} \, \text{d}\hat{x}^i \, \text{d}\hat{x}^j,
\]

\[
M = \int \text{d}^3y \, T_{00}^{\text{eff}},
\]

total effective mass

(including gravitational binding energy)

\[
\mathcal{M} = \int \text{d}^3y \, T_{00}^{\text{eff}},
\]

outside matter

\[
\mathcal{M} = \int \text{d}^3y \, T_{00}^{\text{eff}} = \int \text{d}^3y \, T_{ij}^{\text{eff}} \quad (\nabla^\mu T_{\mu\nu}^{\text{eff}} = 0)
\]

\[
= \int \text{d}S_i \, T_{0i}^{\text{eff}} = - \int \text{d}S_i \, T_{ei}^{\text{eff}} = - \int \text{d}S_i \, T_{0i}^{\text{eff}}.
\]