Recap: TOU equations

\[ ds^2 = -e^{2\Phi} dt^2 + \frac{dr^2}{1 - 2m(r)} + r^2 (d\Omega^2 + \sin^2 \theta d\phi^2) \]

\[ \frac{d\Phi}{dr} = \frac{\mu}{\mu(r)} \quad \frac{d\mu}{dr} = -(\rho + P) \frac{\mu + 4\pi P r^3}{\mu(r)} \quad m(r) = \int_0^r 4\pi \rho \, dr \]

Recovering the Schwarzschild metric outside the star

If matter only extends to \( r > R \) (i.e. \( \rho = P = 0 \) for \( r > R \)), then, for \( r > R \):

\[ m(r) = \frac{M}{2} = \int_0^r 4\pi \rho \, dr \equiv \Pi = \text{constant}. \]

\[ \Phi(r) = \Phi(R) + \int_R^r \frac{M}{r(n-2\mu)} \, dr = \frac{1}{n} \ln \left( \frac{r^2}{2M} \right) + C, \quad C = \text{constant} \]

\[ \text{rescale} \, r \to r - C \Rightarrow ds^2 = -(1 - \frac{2M}{r}) \, dt^2 + \frac{dr^2}{1 - 2M/r} + r^2 \, d\Omega^2. \]

\[ \Rightarrow \text{The metric outside the star becomes the Schwarzschild metric.} \]

Note: \( 4\pi \rho \, dr = \sqrt{1 - \frac{2M}{r}} \, d(\text{proper volume}) \approx \left( 1 - \frac{M}{r} \right) \, d(\text{proper volume}). \)

\[ \Rightarrow \Pi = \int_0^R 4\pi \rho \, dr = \int_0^R \left( 1 - \frac{M}{r} \right) \rho \, d(\text{proper volume}), \quad \text{includes gravitational binding energy} \]

Uniform-density star

\[ \rho = \rho_0 = \text{constant} \, \rho_0, \quad m(r) = \frac{4\pi}{3} \rho_0 r^3 \]

\[ \frac{d\rho}{dr} = -(\rho + P) \frac{4\pi r^2}{3} \frac{\rho_0}{1 - \frac{4\pi P r^3}{3\rho_0}} \quad M = \frac{4\pi}{3} \rho_0 R^3 \]
Rescale variables: $\tilde{\nu} = \frac{\nu}{\rho}$, $\tilde{\eta} = \frac{\eta}{\rho}$

Drop the tildes and get

$$\frac{d\tilde{\nu}}{d\tilde{\eta}} = -\frac{\pi}{R} \frac{1}{1-2\pi \tilde{\eta}} (1+3\tilde{\nu}) (1+3\tilde{\nu})$$

Solution: \[\int_{\tilde{\rho}}^{\nu} \frac{d\tilde{\nu}}{(1+\tilde{\nu})(1+3\tilde{\nu})} = -\frac{\pi}{R} \int_{0}^{2} \frac{d\tilde{\eta}}{1-2\pi \tilde{\eta} \tilde{\eta}^2}, \quad \rho = \text{central pressure.}\]

$$\left[ \ln \left( \frac{1+3\tilde{\nu}}{1+\tilde{\nu}} \right) \right]_{\rho_c}^{\nu} = \frac{1}{2} \ln \left( 1 - \frac{2\pi}{R} \tilde{\eta}^2 \right) \Rightarrow \frac{1+3\tilde{\nu}}{1+\tilde{\nu}} = \frac{1+3\rho_c}{1+\rho_c} \sqrt{1 - \frac{2\pi}{R} \tilde{\eta}^2}$$

\[\Rightarrow \rho = \frac{C \sqrt{1 - \frac{2\pi}{R} \tilde{\eta}^2} - 1}{3 - C \sqrt{1 - \frac{2\pi}{R} \tilde{\eta}^2}} \]

\[C = \frac{1+3\rho_c}{1+\rho_c}\]

Imposing $\rho = 0$ at the surface $\eta = 1$ we get $\frac{1+3\rho_c}{1+\rho_c} = \frac{1}{(1 - \frac{2\pi}{R})^{\frac{1}{2}}}$

\[\Rightarrow \rho_c = \frac{1 - \sqrt{1 - \frac{2\pi}{R}}}{3 - \sqrt{1 - \frac{2\pi}{R}}} \quad \text{diverges for} \quad 3 \sqrt{1 - \frac{2\pi}{R}} \rightarrow 1, \quad \text{i.e.} \quad \frac{\pi}{R} \rightarrow \frac{4}{3}\]

\[\Rightarrow \text{Maximum mass of a constant-density object:} \quad \Pi_{\text{max}} = \frac{4}{3} R.\]

In fact, this remains true for a star with uniformly decreasing density $\frac{d\rho}{d\eta} \leq 0$.

This kind of limit is unique to G.R. (For Newtonian limit, $\rho \approx \frac{2\pi}{3} (R^3 - \eta^2)$)

- Embedding diagrams

Consider manifold of dimension $n$. Metric $g_{\mu\nu}$ has $\frac{(n+1)}{2}$ components. In coordinates $= 0$ can set $n$ components to zero, left with $\frac{(n-1)}{2}$.
We want to embed this manifold in $\mathbb{R}^N$, i.e. describe the manifold by $N-n$ equations $f_1(y', \ldots, y^n) = 0, \ldots, f_{N-n}(y', \ldots, y^n) = 0$, or, equivalently.

Equivalently, manifold is described by expressing $N-n$ coordinates of $\mathbb{R}^N$ as function of the other $n$. Then enforce that $ds^2 = g_{ab} dy^a dy^b = \eta_{\mu\nu} dx^\mu dx^\nu$.

We need at least $n \left( \frac{n-1}{2} \right)$ chooseable functions to match $g_{\mu\nu}$.

$\Rightarrow N-n \geq n \left( \frac{n-1}{2} \right)$, i.e. $N \geq n \left( \frac{n+1}{2} \right)$. For $n = 2$, can hope to embed in $N=3$ (not guaranteed, however!)

Consider the equatorial plane at $t = \text{constant}$. Forms a sub-manifold, metric

\[(2) ds^2 = (1 - \frac{2m}{\rho})^{-1} dn^2 + \rho^2 d\Psi^2 \]

We seek $\varphi(n)$ s.t. $ds^2 = dn^2 + dz^2 + \rho^2 d\Psi^2$ (flat space in cylindrical coords).

$\Rightarrow 1 + \left( \frac{dn}{dh} \right)^2 = \left( \frac{2m}{\rho}\right)^{-1} \Rightarrow \frac{dn}{dh} = \left( \frac{1}{1 - \frac{2m}{\rho}} \right)^{\frac{1}{2}} = \left( \frac{2m}{1 - 2m/\rho} \right)^{\frac{1}{2}}$

For a constant-density star, $\varphi(n) = \Psi \left( \frac{n}{R} \right)^3$ for $n \leq R$ and $\Psi$ for $n > R$.

* for $n \leq R$, \[\frac{dn}{dh} = \left[ \frac{2\rho}{R^3} \right]^{-\frac{1}{2}} = -\sqrt{\frac{R^3}{2\rho}} \frac{dR}{dh} \sqrt{1 - \frac{2M}{R^3}} \]

$\Rightarrow \varphi(n \leq R) = \sqrt{\frac{R^3}{2\rho}} \left( 1 - \sqrt{1 - \frac{2M}{R^3}} \right)$

* for $n > R$, \[\frac{dn}{dh} = \left( \frac{n}{2M} - 1 \right)^{-\frac{1}{2}} = 4\Psi \frac{dR}{dh} \sqrt{\frac{n}{2M} - 1} \]

$\Rightarrow \varphi(n > R) = \varphi(R) + 4\Psi \left( \sqrt{n/2M - 1} - \sqrt{R/2M - 1} \right)$

As $R \to 2M$, $\frac{dn}{dh} \bigg|_R \to \infty$
Kruskal–Szekeres extension for Schwarzschild black hole.

\[ ds^2 = -(1 - \frac{2M}{r}) \, dt^2 + (1 - \frac{2M}{r})^{-1} \, dr^2 + r^2 \, d\Omega^2 \quad \text{explicit everywhere} \]

Consider \( n > 2\pi \) only for now.

We will proceed similarly as for the Rindler spacetime (Homework 5).

Null radial geodesics \((d\Theta = 0 = d\Phi)\):

\[ \frac{dt}{dr} = \pm \left( 1 - \frac{2M}{r} \right)^{-1} \]

\[ t = \pm \left( 2 + 2M \ln \left( \frac{r}{2\pi M} - 1 \right) \right) + \text{constant} \equiv n_* + \text{constant} \]

\( t = t_* \) ("tortoise coordinate")

Define \( u = t - n_* \) \quad \( du = dt - \left( 1 - \frac{2M}{r} \right)^{-1} \, dr \)

\( v = t + n_* \) \quad \( dv = dt + \left( 1 - \frac{2M}{r} \right)^{-1} \, dr \)

\[ ds^2 = -(1 - \frac{2M}{r}) \, du \, dv = -\frac{2M}{r} \, e^{-\frac{n_*}{2M}} \, e^{\frac{u}{4M}} \, du \, dv \]

Define \( U = -e^{-\frac{u}{4M}}, \quad V = e^{\frac{v}{4M}} \) \quad \( ds^2 = -\frac{32M^3}{r} \, e^{-\frac{n_*}{2M}} \, du \, dv + r^2 \, d\Omega^2 \)

Finally, \( T = \frac{V+U}{2}, \quad X = \frac{V-U}{2} > 0 \) \quad \( ds^2 = \frac{32M^3}{r} \, e^{-\frac{n_*}{2M}} \left( -dT^2 + dX^2 \right) + r^2 \, d\Omega^2 \)

Explicit transformation: \( \left( \frac{r}{2M} - 1 \right) \, e^{\frac{n_*}{2M}} = X^2 - T^2 > 0 \)

\[ \frac{T}{2M} = \ln \left( \frac{X+T}{X-T} \right) \]

\( n = n_0 > 2\pi \) \( \Rightarrow \) \( X^2 - T^2 = \text{constant} > 0 \) (hyperbola)

\( n \to 2\pi \) \( \Rightarrow \) \( X^2 - T^2 = 0 \), \( \text{i.e.} \ \ T = \pm X \).

\( t = t_0 = \text{constant} \) \( \Rightarrow \) \( X/T = \text{constant} \).
The function \( f(t) = (\frac{n}{2\pi n} - 1)e^{\frac{2n}{2\pi n}} \) is strictly increasing.

\[ \exists n > 0 \text{ s.t. } (\frac{n}{2\pi n} - 1)e^{\frac{2n}{2\pi n}} = x^2 - T^2, \text{ for } x^2 - T^2 > -1, \text{ i.e. } |T| < \sqrt{1 + x^2} \]

While we initially defined \( X, T \) starting from \( n > 2\pi n \), the metric in \( (X, T) \) coordinates is well-behaved for \( -\infty < x < \infty, \ -\sqrt{1+x^2} < T < \sqrt{1+x^2} \).

It also satisfies the vacuum Einstein field equations.

**Null geodesics:** \( T = \pm x + \text{const} \)

Any radially infalling particle (massive or massless) starting in \( (\#) \), crosses \( n = 2\pi n \) to region \( (\#') \), and eventually reaches the singularity \( n = 0 \) \( (T = \sqrt{1+x^2}) \).

No particle can ever escape region \( (\#') \)
\[ \Rightarrow \text{"Black hole"} \]

\[ T = \sqrt{1+x^2} \Rightarrow dt = \frac{x\,dx}{\sqrt{1+x^2}} \Rightarrow -\sqrt{1+x^2} + dx^2 = \frac{dx^2}{1+x^2} > 0 \]

\[ \Rightarrow \text{The singularity } n = 0 \text{ is a spacelike hypersurface.} \]

Region \( (\#') \) is the "time reversed" of region \( (\#) \): any observer in \( (\#') \) originated from the singularity and escapes to \( (\#) \) in finite proper time. "White hole".

Region \( (\#\#') \) is another asymptotically flat region, that can never be reached from \( (\#) \).

\( (\#) \) and \( (\#') \) are probably purely mathematical and cannot be produced from collapse of some initial matter configuration.