• Conformal transformations (Carroll Appendix G).

A conformal transformation is a local change of scale $\tilde{g}_{\mu \nu} = \omega^2(x) g_{\mu \nu}$ ($\omega > 0$).

* Null, timelike, spacelike characters preserved: some causal structure.

* Angles between non-null vectors are invariant: $\hat{g}_{\mu \nu} \hat{X}^\mu \hat{X}^\nu \tilde{\Xi}^\mu \tilde{\Xi}^\nu$.

* $\tilde{\Gamma}_{\mu \nu}^\alpha - \Gamma_{\mu \nu}^\alpha = C_{\mu \nu}^\alpha$ is a tensor.

Explicitly: $C_{\mu \nu}^\alpha = \frac{1}{\omega} (2 \delta(\mu) \omega - g_{\mu \nu} \nabla_{(\mu} \omega_{)}$.

* Null geodesics of $g_{\mu \nu}$ are also null geodesics of $\tilde{g}_{\mu \nu}$ (but NOT timelike geodesics).

Suppose $\tilde{x}^\nu(\lambda)$ is a null geodesic of $\tilde{g}_{\mu \nu}$. $\tilde{\nabla}_{(\mu} \tilde{X}^{\nu (\lambda)} + C_{\nu \lambda}^{\alpha} \tilde{\nabla}^{\nu \alpha}) = 0$. $\tilde{\nabla}_{(\mu} \tilde{X}^{\nu (\lambda)} = 0$.

Search for $\tilde{x} = 0(\lambda)$, $\tilde{p}^{\mu} = \frac{d\tilde{x}^\mu}{d\lambda} = \tilde{p}^{\mu}$. is null.

$$\tilde{p}^{\mu} \tilde{\nabla}_{(\mu} \tilde{p}^{\nu (\lambda)} = \frac{1}{\omega^2} \tilde{p}^{\mu} \left[ \tilde{\nabla}_{(\mu} \left( \tilde{p}^{\nu (\lambda)} + C_{\nu \lambda}^{\alpha} \tilde{\nabla}^{\nu \alpha} \right) \right] = \frac{2}{\omega^2} \tilde{p}^{\mu} \tilde{p}^{\nu} \nabla_{(\mu} \omega_{)} - \frac{\omega^2}{3} \tilde{p}^{\mu} \tilde{p}^{\nu}$$

$$= \frac{1}{\omega^2} \tilde{p}^{\mu} \left[ \tilde{\nabla}_{(\mu} \left( \tilde{p}^{\nu (\lambda)} + C_{\nu \lambda}^{\alpha} \tilde{\nabla}^{\nu \alpha} \right) \right] = \frac{1}{\omega^2} \left[ \frac{d}{d\lambda} \left( \ln \tilde{p}^{\nu (\lambda)} + \ln \omega^2 \right) \right] \tilde{p}^{\nu (\lambda)}$$

⇒ Set $\frac{d\tilde{x}^{\nu (\lambda)}}{d\lambda} |_{\text{geodesic}} = \frac{d\lambda}{d\lambda} = \omega^2$ and get $\tilde{p}^{\mu} \tilde{\nabla}_{(\mu} \tilde{p}^{\nu (\lambda)} = 0$.

* The Weyl tensor (the completely trace-free part of Riemann) is unchanged $\tilde{C}_{\mu \nu \rho \sigma} = C_{\mu \nu \rho \sigma}$ (note: specifically with this placement of indices).

(but the Ricci tensor is different).
• **Conformal (or Penrose) diagrams**

  Goal: visual representation of (infinite) spacetime with a finite coordinate range, and preserving causal structure.

  • **Minkowski spacetime**; spherical polar coords: $ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$. $r > 0$
    * define $u = t - r$, $v = t + r$ $-\infty < u < v < \infty$ $ds^2 = -du dv + \frac{(u-v)^2}{4} d\Omega^2$
    * define $U = \arctan(u)$, $V = \arctan(v)$ $-\frac{\pi}{2} < U < V < \frac{\pi}{2}$.
    
    $dudv = (\cos U \cos V)^2 dU dV$
    
    $v-u = \tan V - \tan U = \frac{\sin V \cos U - \sin U \cos V}{\cos U \cos V} = \frac{\sin (V-U)}{\cos U \cos V}$

  $\Rightarrow$ $ds^2 = (\cos U \cos V)^2 [- dU dV + \frac{1}{4} \sin^2 (V-U) d\Omega^2]$

  * define $T = V+U$, $R = V-U$ $0 < R < \pi$ $-\pi < R + T < \pi$ $\Rightarrow \pi < T < \pi - R$
    
    $\cos U \cos V = \frac{1}{2} (\cos (V+U) + \cos (V-U)) = \frac{1}{2} (\cos T + \cos R) \equiv \frac{1}{2} \omega$

    $$ds^2 = \omega^2 (T, R) \left[ -dT^2 + dR^2 + \sin^2 R d\Omega^2 \right] \equiv \omega^2 (T, R) ds^2.$$  

  $\Rightarrow$ Minkowski spacetime is conformally related (hence has the same causal structure) to the (unphysical) spacetime with metric
    $$ds^2 = -dT^2 + dR^2 + \sin^2 R \left( d\Omega^2 + \sin^2 \Theta d\varphi^2 \right)$$

  metric on a 3-sphere $S^3$.

  $0 < R < \pi$, $0 < \Theta < \pi$, $0 < \varphi < 2\pi$.

  I.e. Minkowski is conformally related to (part of) $R \times S^3$

  (Full transformation: $T = \arctan(t+1) + \arctan(t-1)$, $R = \arctan(t+1) - \arctan(t-1)$)
Each line $T = \text{constant}$ represents a 3-sphere. Radial light cones ($d\theta = d\varphi = 0$) have 45° angles.

- $i^\pm$ $t = \text{constant}$, $r \to \pm \infty$: "future ($+$) and past ($-$) timelike infinity."
- $i_0^\pm$ $t = \text{constant}$, $r \to +\infty$: "spatial infinity." 
- $j^\pm$ "future ($+$) and past ($-$) null infinity" (pronounced "scri plus," "scri minus").

$i^0\pm$ are points, $j^\pm$ are null surfaces.

Timelike geodesics start at $i^-$ and end at $i^+$. All outgoing null geodesics end at $j^+$. All incoming null geodesics started at $j^-$.  

- *Schwarzschild spacetime.*

\[ ds^2 = \frac{32\ M^3}{\ell^2} \ e^{-\frac{r}{2M}} (-dT^2 + d\chi^2) + \ell^2 dS^2 \]
\[ \left( \frac{\ell}{2M} - 1 \right) e^{\frac{\ell}{2M}} = \chi^2 - T^2 \geq -1 \]
\[ \frac{\ell}{2M} = \ln \left( \frac{\chi + T}{\chi - T} \right) \]

Same procedure. $\tilde{T} = \text{arctan} \left( T + \chi \right) + \text{arctan} \left( T - \chi \right)$
\[ X = \text{arctan} \left( T + \chi \right) - \text{arctan} \left( T - \chi \right). \]
If it weren't for the restriction \( |T| < \sqrt{1 + x^2} \) (corresponding to \( r > 0 \)), conformal diagram would be a diamond (because \( x \), the equivalent of \( n \) in the Minkowski case, runs from \(-\infty \) to \( +\infty \)).

\[
\tan (a + s) = \frac{\tan(a) + \tan(s)}{1 - \tan(a)\tan(s)}
\]

\[
\Rightarrow \tan \frac{\pi}{2} = \frac{2T}{1 + x^2 - T^2} \quad \text{as} \quad T \to \pm \sqrt{1 + x^2}
\]

\[
\Rightarrow \frac{\pi}{2} \quad \text{as} \quad T \to \pm \sqrt{1 + x^2}
\]

This represents the singularity \( n = 0 \)

Advantage of this diagram: compact representation of all spacetime (well, 2 dimensions of it). Radial null geodesics (i.e. with \( d\Theta = d\Phi = 0 \)) are at \( 45^\circ \).

The asymptotic structure is similar to flat spacetime.

Note: it is not part of \( n = 0 \).

**Collapsing star**

As long as the star covers the region \( n < 2\pi \), the spacetime is similar to Minkowski (so \( n = 0 \) is vertical axis).

Once stellar radius \( < 2\pi \), the spacetime becomes that of a BH.

But we only expect regions \( \Box \) and \( \Box \), no white hole or other asymptotically flat region.