Ken black holes. [Tromb & Blandford Chapter 26]

Ken metric in Boyer–Lindquist coordinates: (only $a < M$ otherwise naked singularity)

$$
\begin{align*}
\text{ds}^2 &= -\frac{\rho^2}{\Delta} \, dt^2 + \frac{\rho^2}{\Delta} \, dr^2 + \rho^2 \, d\theta^2 + \frac{\rho^2}{\beta^2} \sin^2 \theta \left( d\Phi - \omega \, dt \right)^2 \\
\Delta &= r^2 - 2Mr + a^2 \\
\beta^2 &= r^2 + a^2 \cos^2 \theta \\
\Xi &= (r^2 + a^2)^2 - a^4 \sin^2 \theta \\
\omega &= \frac{2Mr}{\Xi}
\end{align*}
$$

Inverse metric:

$$
\begin{align*}
\text{g}^{-1} &= -\frac{\Xi^2}{\rho^2 \beta^2} \left( \partial_t + \omega \partial_\Phi \right)^2 + \frac{\Delta}{\beta^2} \partial_r^2 + \frac{1}{\rho^2} \partial_\theta^2 + \frac{\rho^2}{\Xi^2 \sin^2 \theta} \partial_\Phi^2
\end{align*}
$$

* For $a \to 0$, reduces to Schwarzschild.

* For $r \gg M, \alpha$, 

$$
\text{ds}^2 \approx -\frac{\Xi^2}{2} \left( \partial_t + \omega \partial_\Phi \right)^2 + \frac{\Delta}{\beta^2} \partial_r^2 + \frac{1}{\rho^2} \partial_\theta^2 + \frac{\rho^2}{\Xi^2 \sin^2 \theta} \partial_\Phi^2
$$

$\Rightarrow$ The parameter $\alpha$ is the angular momentum per mass, $\alpha = \frac{J}{M}$.

* Two Killing vector fields: $\frac{\partial}{\partial t}$ (stationary, but not static because $g_{tt} < 0$) and $\frac{\partial}{\partial \phi}$

$\Rightarrow$ $p_t$ and $p_{\phi}$ are conserved along geodesics.

Suppose a particle falls from infinity, starting at rest: $u_t|_{\infty} = -u_t|_{\infty} = -1$

$$
\begin{align*}
\frac{d\Phi}{dt} &= \frac{u^\Phi}{u^t} = \frac{g^{tt} u_t + g^{t\Phi} u_\Phi}{g^{tt} u_t + g^{t\Phi} u_\Phi} = \frac{g^{tt}}{g^{tt}} = \omega = \frac{2Mr}{\Xi}
\end{align*}
$$

$\Rightarrow$ as the particle falls, it acquires an angular velocity $\omega$. Frame dragging.

Note: This statement is only meaningful because $\partial_\Phi$ and $\partial_t$ are "special" coordinates, tied to the symmetries of spacetime.
• **Light-cone structure**

* \( \Sigma^2 \geq (n^2 + a^2)^2 - a^2(n^2 - 2Mn + a^2) = n^4 + n^2a^2 + 2a^2Mn > 0 \).

* \( D = n^2 - 2Mn + a^2 = 0 \) at \( n = \pm \sqrt{M^2 - a^2} \).

* Consider null geodesics at \( n > n_+ \) (so \( D > 0 \)):

  \[
  \frac{d^2\ell}{dt^2} + \frac{\Sigma^2}{\rho^4} \sin^2 \theta \left( \frac{d\theta}{dt} - \omega \right)^2 + D \left( \frac{d\theta}{dt} \right)^2 = \frac{\Delta}{\Sigma^2}
  \]

  Innermost and outermost edges of the cone: where \( \frac{d\theta}{dt} \) maximized, i.e.
  \[
  \frac{d\theta}{dt} = 0, \quad \frac{d\ell}{dt} = \omega, \quad \frac{dn}{dt} = \pm \frac{\Delta}{\Sigma^2}.
  \]

  \( \Rightarrow \) The light cone "pinches off" and "closes" at \( n \to n_+ \) (\( = 2\pi \) for \( a = 0 \)).

This is similar to the Schwarzschild case: a particle falling towards the black hole never appears to cross the horizon at \( n_+ = \pi + \sqrt{M^2 - a^2} \), and seems to spiral around "forever" (in Boyer-Lindquist time).

But it takes finite proper time (like in Schwarzschild) to reach \( n_+ \).

Also, tides are finite at \( n_+ \) \( \Rightarrow \) coordinate singularity.

One can show, like in Schwarzschild, that nothing special happens at \( n_+ \) but that no particle can escape from inside \( n_+ \).
Extraction of rotational energy (Penrose process).

After some algebra, \( g_{tt} = \frac{\rho^2}{\Sigma^2} \delta + \frac{\Sigma^2}{\rho^2} a^2 \sin^2 \theta = -\frac{1}{\rho^2} (\rho^2 - \Sigma^2 + a^2 \cos^2 \theta) \)

\( \Rightarrow g_{tt} \) becomes positive for \( \Sigma < \Sigma_{\text{ergo}} = 1 + \sqrt{\rho^2 - a^2 \cos^2 \theta} \)

Inside the ergosphere, all particle must have \( \frac{dv}{dt} > 0 \).

Suppose one drops a particle from \( \infty \). \( E_{\infty} = -p_t \) conserved.

Particle decays to 2 particles inside the ergosphere, one plunging into the BH, the other one going back to \( \infty \). \( E_{\text{plunge}} = E_{\text{out}} + E_{\text{out}} = -p_t - p_t \)

Since \( \frac{2}{3} \) is spacelike inside the ergosphere, and \( p_{\text{plunge}} \) is timelike, it is in principle possible to arrange \( E_{\infty} < 0 \), hence extract energy.

This is, of course, a highly idealized set up...
Introduction to cosmology (based on C. Hirota's lecture notes).

Basic assumption (with observational evidence!): The Universe is approximately homogeneous and isotropic. First step is to consider exactly homogeneous and isotropic solutions and then study perturbations.

Spacetime can be sliced into homogeneous and isotropic 3-surfaces $\Sigma_t$, labeled by a time coordinate $t$. There exist "comoving observers" who see an isotropic Universe, whose characteristic only depend on $t$.

Spatial coordinates $x^i$ on each $\Sigma_t$, labeling comoving observers.

* No preferred direction $\Rightarrow g_{0i} = 0$.

General metric: $ds^2 = -g_{tt} dt^2 + g_{ij} dx^i dx^j$

Comoving observers have 4-velocity $u^0 = (-g_{tt})^{1/2}$, $u^i = 0$.

* Spatial components of 4-acceleration of comoving observers must vanish (else, it would be a preferred direction). Note: This implies $a^0 = 0$.

$$a^i = u^k \nabla_k u^i = u^k (\partial_k u^i + \Gamma^i_{jk} u^j) = -\frac{1}{2} \Gamma^i_{00} = -\frac{1}{2} g_{00} \partial_t g_{tt}$$

$$a^i = 0 \Rightarrow \partial_t g_{tt} = 0$$. So $g_{tt}$ depends only on time.

Rescale $t \Rightarrow ds^2 = -dt^2 + g_{ij} dx^i dx^j$, $u^0 = 1$, $u^i = 0$

* Isotropy of local expansion: $\nabla^i u_i = H(t) \delta^i_0$, $H$ some function of $t$ only.

$$\nabla^i u_i = \partial_t u^i + \Gamma^i_{00} = \frac{1}{2} g^{ik} \partial_k g_{tt} \Rightarrow \partial_t g_{tt} = 2 H(t) g_{tt}$$

Define $a = e^{\int H(t) dt}$, i.e. $H = \frac{\partial a}{a} \Rightarrow g_{ij} = a^2(t) \delta^i_j(x^l)$.

$$ds^2 = -dt^2 + a^2(t) \delta^i_j(x^l) dx^i dx^j$$

H: Hubble rate

a: scale factor

Friedmann - Lemaître - Robertson - Walker (FLRW) metric.