Kinetic theory

The phase-space density $N$ is the number of particles per phase-space volume element $d\xi^0 d\xi^1 d\xi^2 d\xi^3 dp_x dp_y dp_z$.

$$N = \frac{dN}{d\xi_x d\xi_y d\xi_z}$$

This is Lorentz invariant.

- $d\xi^0 d\xi^1 d\xi^2 d\xi^3$ is Lorentz invariant ($\det A = 1$).
- $d\xi^0$ is Lorentz invariant.
- $p^0 = m \frac{dx^0}{d\xi^0}$ implies $p^0 d\xi_x$ is Lorentz invariant.
- $dp_x dp_y dp_z \int dp_x \int dp_y \int dp_z d\epsilon_d \left( \epsilon_{\rho} p^\rho + m^2 \right) = \frac{dN_p}{p^0}$ is Lorentz invariant.

Liouville's theorem: $d\Sigma_x d\Sigma_y$ have $N$ are constant along particle's trajectory in the absence of collision.

It is easy to see if particles move on straight lines.

More generally: if particles satisfy Hamilton's equations:

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x^i} \quad \Rightarrow \quad x^i(t+\Delta t) = x^i(t) + \Delta t \frac{\partial H}{\partial p_i} \equiv x^i'$$

$$p_i(t+\Delta t) = p_i(t) - \Delta t \frac{\partial H}{\partial x^i} \equiv p_i'$$

$$d\Sigma_x d\Sigma_y = \left| \det \begin{pmatrix} \frac{\partial \epsilon'}{\partial x} & \frac{\partial \epsilon'}{\partial p} \\ \frac{\partial \epsilon}{\partial x} & \frac{\partial \epsilon}{\partial p} \end{pmatrix} \right| d\Sigma_x d\Sigma_y$$

$$= \left| \det \left[ \prod_{i=1}^6 + \Delta t \begin{pmatrix} \frac{\partial^2 H}{\partial p_i \partial x^j} & \frac{\partial^2 H}{\partial p_i \partial p_j} \\ -\frac{\partial^2 H}{\partial x^i \partial p_j} & -\frac{\partial^2 H}{\partial x^i \partial p_j} \end{pmatrix} \right] \right| d\Sigma_x d\Sigma_y$$
\[ \text{det} \left[ 1 + \Delta \Pi \right] = 1 + \Delta \text{trace} (\Pi) + O (\Delta^2). \]

\[ \text{trace} (\Pi) = \frac{\partial^4 H}{\partial x^i \partial p_i} - \frac{\partial^4 H}{\partial p_i \partial p_i} = 0. \]

\[ \Rightarrow \Pi_{x}, \Pi_{p} = \Pi_{x} d\Pi_{p} (1 + O (\Delta^2)). \]

\[ \Rightarrow \text{Liouville's equation \text{ [collisionsless Boltzmann equation]}}: \]

\[ 0 = \frac{dN}{dt} \bigg|_{\text{hyp}} = \frac{dx^m}{dt} \frac{\partial N}{\partial x^m} + \frac{dp_n}{dt} \frac{\partial N}{\partial p_n}. \]

Stress-energy momentum tensor: \[ T^{\mu \nu} = \int dV_{p} \ N \ p^{\mu} \frac{dx^m}{dt} = \int dV_{p} \ N \ p^{\mu} p^{\nu}. \]

Current density: \[ J^{\mu} = q \int \frac{dV_{p}}{\rho^{0}} \ N \ p^{\mu}. \] (if particles all have charge \( q \)).

Exercise: show that \( \partial_{\mu} T^{\mu \nu} = F^{\mu \nu} J_{\nu} \)

- Gravity as a special-relativistic force?

\[ \frac{d\vec{\Phi}}{dt} = -m \vec{\nabla} \Phi \quad \Rightarrow \text{maybe} \quad \frac{d\Phi}{dt} = -m \partial_\alpha \Phi? \]

Hypoth: \[ \frac{d (m^2)}{dt^2} = - \frac{d}{dt} (p^{\mu} \partial_{\mu} \Phi) = -2 \ p^{\mu} \frac{df_{\Phi}}{dt} = -2 \ m \ p^{\mu} \partial_{\mu} \Phi \]

\[ \Rightarrow \text{in a frame where} \ \Phi = 0, \ \text{get} \ \frac{d\Phi}{dt} = -\Phi \quad \times \]

How about \[ \frac{d(p^{\mu})}{dt^2} = -m \left( \eta^{\mu \nu} + u^\mu u^\nu \right) \partial_{\nu} \Phi? \]

\[ \Rightarrow \frac{d}{dt} (e^{\vec{\Phi} \ p^{\mu}}) = e^{\vec{\Phi}} \left( u^\mu \partial_{\nu} p^{\nu} - m \eta^{\mu \nu} \partial_{\nu} \Phi - m u^\nu \partial_{\nu} \Phi \right) \]

\[ = -m e^{\vec{\Phi}} \eta^{\mu \nu} \partial_{\nu} \Phi \]

\[ \Rightarrow \text{For massless particles}, \ e^{\vec{\Phi} \ p^{\mu}} \text{ constant} \Rightarrow \text{no gravitational lensing}. \quad \times \]
The equivalence principle

\[ \text{min} \int \frac{d^2x'}{d\tau^2} = m \text{grav} g \quad \text{"weak equivalence principle" :} \quad \text{min} \int \frac{d^2x'}{d\tau^2} = m \]

Suppose we define \( x' = x - \frac{1}{2} g t^2 \) \( \Rightarrow \) \( m \frac{d^2x'}{dt^2} = 0 \)

\( t' = t \)

⇒ The motion of freely-falling particles is the same in a gravitational field and in a uniformly accelerated frame, in small enough region of spacetime.

⇒ It is impossible to measure local gravitational fields by observing free-falling particles: they all have the same "gravitational change".

In contrast, we can measure local electromagnetic fields by observing the motion of charged vs. neutral particles.

• Strong equivalence principle: in small enough regions of spacetime, it is always possible to choose a locally inertial frame/grounds in which all the laws of nature take the form of special relativity without gravity.

Carroll defines the "Einstein equivalence principle" as applying to only non-gravitational laws. For instance, it could have been that the gravitational binding energy gets a different mineral and weight.

⇒ It is impossible to measure local gravitational fields by means of any experiment.

"small enough", "local": ie small enough that the "gravitational field" is approximately uniform.

If the region is large enough, then we can observe tides.
Gravitational redshift: a direct consequence of the E.P.

Setup (1): constant z

Emitter accelerating at constant \( \ddot{a} \to \)

Receive

Both initially at rest. Emitter sends photon of energy \( E_{em} \)

For \( z \) small enough, photon received at \( r = z \).

By then, receiver has velocity \( v = at = az \). Suppose \( v \ll c \).

Observed photon energy \( E_{obs} = E_{em} (1-v) \), i.e. \( \frac{\Delta E}{E} = -v = -az \).

Setup (2): undistinguishable (E.P)

\( \Delta E \)

Emitter Undistinguishable (E.P)

\( \Delta E = -az = g = \Phi(\text{em}) - \Phi(\text{em}) \).
• Geodesic equation.

Consider a particle freely falling, i.e., under the influence of gravity only.

\[ E.P. \Rightarrow \text{coordinates } S^k \text{ s.t. } \frac{d^2 S^k}{dt^2} = 0, \quad ds^2 = -\eta_{\mu\nu} ds^\mu ds^\nu. \]

Let's rewrite this in an arbitrary coord. system \( x^\mu. \)

\[ 0 = \frac{d}{dt} \left( \frac{\partial S^\mu}{\partial x^\nu} dx^\nu \right) = \frac{\partial^2 S^\mu}{\partial x^\nu \partial x^\rho} \frac{dx^\rho}{dt} + \frac{\partial S^\mu}{\partial x^\nu} \frac{dx^\nu}{dt} \frac{dx^\rho}{dt} \frac{dx^\rho}{dt} \left( \times \frac{\partial x^\sigma}{\partial S^\rho} \right) \]

\[ \Rightarrow \quad \frac{\partial^2 S^\mu}{\partial t^2} + \nabla^\nu \frac{\partial S^\mu}{\partial x^\nu} \frac{dx^\nu}{dt} \frac{dx^\nu}{dt} = 0 \]

\[ ds^2 = -\eta_{\mu\nu} \frac{\partial S^\mu}{\partial x^\lambda} \frac{\partial S^\nu}{\partial x^\rho} dx^\lambda dx^\rho = -g_{\mu\nu} dx^\mu dx^\nu, \quad g_{\mu\nu} = \frac{\partial S^\mu}{\partial x^\lambda} \frac{\partial S^\nu}{\partial x^\rho} \eta_{\lambda\rho}. \]

• Determining the local inertial frame.

Suppose we are given \( \eta_{\mu\nu} \) and \( \Gamma^\lambda_{\mu\nu}. \Rightarrow \frac{\partial^2 S^\mu}{\partial x^\nu \partial x^\rho} = \nabla^\nu \frac{\partial S^\mu}{\partial x^\rho} \]

\[ \Rightarrow \quad S^\mu (x) = \alpha^\mu + b^\mu \nu (x^\nu - x^\nu) + \frac{1}{2} \nabla^\nu \nabla^\rho b^\mu_{\sigma} (x^\nu - x^\nu)(x^\rho - x^\rho) + G(x, x)^3. \]

\[ b^\mu_{\nu} = \frac{\partial S^\mu}{\partial x^\nu} \quad \text{s.t.} \quad \eta_{\mu\nu} b^\mu_{\nu} = \delta_{\mu\nu}. \quad 16 \text{ unknowns, } b^\mu_{\nu}. \]

This determines the \( b^\mu_{\nu} \) up to a Lorentz transformation. \( b^\mu_{\nu} = \Lambda^\mu_{\rho} b^\rho_{\nu}. \)

\[ \Rightarrow \text{given } \eta_{\mu\nu} \text{ and } \Gamma^\lambda_{\mu\nu}, \text{ the local inertial frame is determined up to an inhomogeneous (\textbf{not}) Lorentz transformation}. \]
• Relation between $\Gamma^\mu_{\nu\lambda}$ and $g_{\mu\nu}$.

From the definition given above, show (exercise) that

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \Gamma^\nu_{\lambda\mu} g_{\rho\nu} + \Gamma^\nu_{\rho\mu} g_{\rho\nu}$$

$$\Rightarrow \Gamma^\nu_{\mu\nu} = \frac{1}{2} g^{\nu\lambda} \left( \frac{\partial g_{\mu\lambda}}{\partial x^\nu} + \frac{\partial g_{\nu\lambda}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda} \right).$$

• Newtonian limit: slow-motion, quasi-static, quasi-Minkowski:

If $v = \frac{d\vec{x}}{dt} \ll 1 \Rightarrow \frac{d^2\vec{x}}{dt^2} + \Gamma^\mu_{\nu\mu} \left( \frac{dr}{dt} \right)^2 = Gf(r)$. (slow motion)

$$\Gamma^\mu_{\nu\mu} = \frac{1}{2} g^{\mu\nu} \left( \frac{\partial g_{\nu\lambda}}{\partial r} + \frac{\partial g_{\nu\nu}}{\partial r} - \frac{\partial g_{\nu\mu}}{\partial r} \right) \approx - \frac{1}{2} g^{\nu\nu} \frac{\partial g_{\nu\nu}}{\partial r} \quad \text{(quasi-static)}$$

quasi-Minkowski: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$; $|h_{\mu\nu}| \ll 1 \Rightarrow \Gamma^\mu_{\nu\mu} \approx - \frac{1}{2} g^{\nu\nu} \frac{\partial h_{\nu\nu}}{\partial r}.$

$$\Rightarrow \left\{ \begin{array}{l}
\frac{d^2\vec{x}}{dt^2} = 0 \Rightarrow \frac{d\vec{x}}{dt} = \text{const} \\ \ \\
\frac{d^2\vec{x}}{dt^2} \approx \frac{1}{2} \nabla \cdot \nabla h_{\nu\nu} \left( \frac{dr}{dt} \right)^2 \Rightarrow \frac{d^2\vec{x}}{dt^2} \approx \frac{1}{2} \nabla \cdot \nabla h_{\nu\nu} \equiv - \nabla^2 \phi
\end{array} \right.$$  

$$\Rightarrow h_{\nu\nu} = -2\phi,$$ where $\phi$ is the Newtonian potential.