Last time we saw that we can always set $g_{\mu\nu} = \delta_{\mu\nu}$ and $\partial_{\mu} g_{\nu\rho} = 0$ (exactly the right number of free functions).

In general, we cannot impose $\frac{d^2 g_{\mu\nu}}{dx^\lambda dx^\lambda} = 0$; there are too few coordinate degrees of freedom:

$$\frac{d^2 g_{\mu\nu}}{dx^\lambda dx^\lambda} : \left(\frac{n(n+1)}{2}\right)^2 \text{ numbers}, \quad \frac{d^3 x^\lambda}{dx^\mu dx^\nu dx^\lambda} : n \times \frac{n(n+1)(n+2)}{6} \text{ numbers}.$$

Too few by $(\frac{n(n+1)}{2})^2 - n^2 \frac{n(n+1)(n+2)}{6} = \frac{n^2(n^2-1)}{12}$. We'll see this is the # of indep. components of Riemann.

**Tangent to a curve.**

$V = \frac{d}{dx}$ is the tangent to the curve $p(x)$.\[\text{i.e. } V p(t) = \frac{dp}{dx} |_p.\]

In a coord. system, $V^\mu = \frac{dx^\mu}{dx}.\]

**Parallel transport.**

- If $M = \mathbb{R}^n$, there is a well-defined way to compare vectors defined at different points: pick a local cartesian coordinate system, and simply compare components of vectors in that system.

- We do have an intuition of what it means to parallel transport a tangent vector along a path on a sphere, for some specific cases:

We want to formalize this to arbitrary manifolds and paths: if $V$ is a vector field, and $T$ is the tangent to a curve $T = \frac{d}{dx}$, "$V$ is parallel transported along the curve" $T \cdot V = 0$.
Covariant derivatives

We already introduced gradients of functions as 1-forms (dual vectors):
\[ df \mid_p (v) = V(f) \mid_p (b). \]  This defines a dual vector field for any function \( f \).

A scalar field \( f \) is constant along a curve \( p(\lambda) \) if \( \frac{df}{d\lambda} = 0 \), i.e. \( df(\frac{d}{d\lambda}) = 0 \).

We want to define the linear operator \( \bigtriangledown \), s.t. if \( T \) is a tensor field of rank \( (k, l) \), \( \bigtriangledown T \) is a tensor field of rank \( (k, l+1) \).

Notation: \( \bigtriangledown_{\alpha} T^{\beta_{\delta}} = (\bigtriangledown T)^{\beta_{\delta}} \bigtriangledown_{\alpha} \equiv T^{\beta_{\delta}} \bigtriangledown_{\alpha} \).

Requirements:

1) For a scalar (i.e., function \( f \)), \( \bigtriangledown f \equiv df \) is the usual gradient.
   \[ \Rightarrow \text{If } \mathbf{V} \text{ is a vector, } \mathbf{V}(f) \equiv \bigtriangledown f(\mathbf{V}) = (\mathbf{V} f) \mathbf{v} = \mathbf{v} \cdot \bigtriangledown f \quad \text{[no coord basis required].} \]

2) Leibniz rule: \( \bigtriangledown (T \otimes S) = T \otimes \bigtriangledown S + \bigtriangledown T \otimes S \)

3) Commutes with contraction e.g. \( (\bigtriangledown T)^{\alpha} \delta_{\beta} = \bigtriangledown T^{\alpha} \delta_{\beta} \), then
   \[ \bigtriangledown_{\alpha} (T^{\beta})^{\delta} = \bigtriangledown T^{\alpha} \delta_{\beta} \quad \text{[i.e. } \bigtriangledown \text{ (contraction } T) = \text{ contraction (}\bigtriangledown T)] \]

4) Torsion-free: \( \bigtriangledown_{\alpha} \bigtriangledown_{\beta} b = \bigtriangledown_{\beta} \bigtriangledown_{\alpha} b \) if \( b \) is a scalar.

Commutator: \[ [X,Y](f) = X(Y(f)) - Y(X(f)) \]
\[ = X^\alpha \bigtriangledown_{\alpha} (Y^\beta \bigtriangledown_{\beta} f) - Y^\alpha \bigtriangledown_{\alpha} (X^\beta \bigtriangledown_{\beta} f) \quad \text{[condition 1 used twice]} \]
\[ = X^\alpha (\bigtriangledown_{\beta} Y^\beta) \bigtriangledown_{\alpha} f + Y^\alpha \bigtriangledown_{\alpha} (X^\beta \bigtriangledown_{\beta} f) - Y^\alpha (\bigtriangledown_{\alpha} X^\beta) \bigtriangledown_{\beta} f + X^\beta \bigtriangledown_{\beta} X^\alpha \bigtriangledown_{\alpha} f \quad \text{[Leibniz]} \]
\[ = X^\beta \bigtriangledown_{\beta} Y^\alpha \bigtriangledown_{\alpha} f - Y^\alpha \bigtriangledown_{\alpha} X^\beta \bigtriangledown_{\beta} f \quad \text{[Torsion-free]} \]
\[ = (X^\beta \bigtriangledown_{\beta} Y^\alpha - Y^\beta \bigtriangledown_{\beta} X^\alpha) \bigtriangledown_{\alpha} f \quad \text{[dummy indices]} \]
\[ \Rightarrow [X,Y]^\alpha = X^\beta \bigtriangledown_{\beta} Y^\alpha - Y^\beta \bigtriangledown_{\beta} X^\alpha \quad \text{(geometric, basis indices).} \]
• Example of covariant derivative: ordinary derivative attached to a coord. system.

Given a coordinate system /chart \( \phi \), define \( \nabla^{(\phi)}_{\alpha} T \) whose components, in the coordinate basis \@& dual basis, are \( \nabla_{\alpha}^{(\phi)} T_{\beta}^{\gamma} \).

This is a tensor, satisfying Leibniz rule, etc., but its components in a different coord. basis \( \phi' \) are NOT \( \nabla_{\alpha}^{(\phi')} T_{\beta}^{\gamma} \).

• Theorem: The difference between 2 covariant derivatives in a tensor field.

Meaning: If \( \nabla \) and \( \tilde{\nabla} \) are covariant derivatives, then \( \exists \) tensor field \( \Gamma^x_{\beta \gamma} \) s.t. \( \nabla_{\alpha} V^x - \tilde{\nabla}_{\alpha} V^x = \Gamma^x_{\beta \gamma} V^\beta \). You will prove this in HW.

⇒ Given a coord. system, \( \nabla \equiv \nabla^{(\phi)} \) is a covariant derivative, so, for any covariant derivative, \( \nabla_{\alpha} V^x = \Gamma^{(\phi) x}_{\alpha \beta} V^\beta + \Gamma^x_{\alpha \beta} V^\beta \).

\( \Gamma^x_{\beta \gamma} \) is a tensor field, BUT, is associated with specific chart \( \phi \) (and \( \nabla \)).

⇒ If \( \tilde{\nabla} \) use a different chart \( \tilde{\phi} \): \( \nabla_{\alpha} V^x = \Gamma^{(\tilde{\phi}) x}_{\alpha \beta} V^\beta + \Gamma^x_{\alpha \beta} V^\beta \)

\( \Gamma^x_{\beta \gamma} \): Christoffel symbol / connection coefficient.

Associated with a particular coordinate basis (and covariant derivative).

\[ \nabla_{\alpha} (V^x U_\alpha) - \tilde{\nabla}_{\alpha} (V^x U_\alpha) = 0 \quad \text{[\( \nabla b \equiv db = \tilde{\nabla} b \) if \( b \) scalar]} \]

\[ = (\nabla_{\alpha} V^x - \tilde{\nabla}_{\alpha} V^x) U_\alpha + V^x (\nabla_{\alpha} U_\alpha - \tilde{\nabla}_{\alpha} U_\alpha) \quad \text{[Leibniz]} \]

\[ = \Gamma^x_{\beta \gamma} V^\beta U_\alpha + V^x (\nabla_{\alpha} U_\alpha - \tilde{\nabla}_{\alpha} U_\alpha) = V^x \left[ \Gamma^x_{\beta \gamma} U_\beta + \nabla_{\alpha} U_\alpha - \tilde{\nabla}_{\alpha} U_\alpha \right] \]

\[ \Rightarrow \nabla_{\alpha} U_\alpha - \tilde{\nabla}_{\alpha} U_\alpha = - \Gamma^x_{\beta \gamma} U_\gamma \]

Take \( U_\alpha = U_{\alpha b} = \tilde{U}_{\alpha b} \) . Torsion-free \( \Rightarrow \) \( \Gamma^x_{\alpha \beta} = \Gamma^x_{\beta \alpha} \).
Exercise: Show that for a rank (0,2) tensor \( g_{\alpha \beta} \),
\[
\nabla_\alpha g_{\beta \gamma} = C^\gamma_{\alpha \beta} g_{\delta \gamma} - C^\gamma_{\alpha \delta} g_{\beta \gamma}.
\]
Generalize to arbitrary ranks.

Recap: There exist covariant derivatives: e.g. partial derivative in a specific coordinate system.

- Given a covariant derivative \( \nabla \), we can build infinitely many such operators: given a tensor field \( C^{\rho \sigma} \), symmetric in \( \beta \delta \),
  \[
  \nabla_\alpha V^\sigma = \nabla_\alpha V^\sigma + C^\sigma_{\alpha \delta} V^\delta,
  \]
  and generalize to arbitrary rank.

At last, given a metric, we may define THE \textbf{metric-compatible} derivative, such that \( \nabla g = 0 \). \( \nabla g = 0 \) preserves the notion of length, norms.

\[
0 = \nabla_\mu g_{\lambda \sigma} = \nabla_\lambda g_{\mu \sigma} = \Gamma^\nu_{\lambda \mu} g_{\nu \sigma} - \Gamma^\nu_{\mu \sigma} g_{\lambda \nu},
\]
(taking \( \nabla = \partial \) in some coordinate basis)

Combine \( \Rightarrow \)

\[
\Gamma^\nu_{\mu \lambda} = \frac{1}{2} g^{\nu \rho} (\partial_\mu g_{\lambda \rho} + \partial_\lambda g_{\mu \rho} - \partial_\rho g_{\mu \lambda})
\]

One of the few equations you may want to learn by heart.

\[ \Rightarrow \text{metric compatibility} \] (and torsion-free) uniquely determines \( \Gamma^\nu_{\mu \lambda} \) become \( \nabla \).

Note: \( \nabla g' = 0 \) as well. Exercise: show that \( \nabla \epsilon = 0 \)

Parallel Transport: A vector field \( V \) is said to be \textit{parallel-transported} along a curve with tangent vector \( T = \frac{d}{d\lambda} \) if \( T \cdot \nabla V = 0 \).

\text{Parallel Transport} \quad \text{along a curve with tangent vector} \quad T = \frac{d}{d\lambda} \text{ if} \quad T \cdot \nabla V = 0

\[
T \cdot \nabla V \quad \text{sometimes denoted} \quad \nabla_T V \text{ or} \quad \frac{DV}{d\lambda}.
\]

\[
\left( \frac{DV}{d\lambda} \right)^\lambda = T^\nu \nabla_\nu V^\lambda = \frac{d}{d\lambda} \left( \partial_\nu V^\lambda + \Gamma^\lambda_{\nu \sigma} V^\sigma \right)
\]
\[ \text{[in a } g\text{-covd. system]} \]
\[
= \frac{dV^\lambda}{d\lambda} + \Gamma^\lambda_{\nu \sigma} \frac{dx^\nu}{d\lambda} V^\sigma
\]