• Flat spacetime $\iff$ Riemann $= 0$

⇒ is clear: for flat spacetime, a global coordinate in which $g_{\mu\nu} = \gamma_{\mu\nu}$.

Let's prove that if Riemann $= 0$, then spacetime is flat.

At a point $p$, find an orthonormal basis of $V_p^*$, $\omega^{(x)}$, i.e., s.t. $g^{|p} = \gamma_{\mu\nu} \omega^{(x)} \otimes \omega^{(x)}$.

Define the one-form fields $\omega^{(x)}$ by parallel-transporting $\omega^{(x)}|_p$.

Since Riemann $= 0$, this is independent of the chosen path, and well-defined.

Since $\omega^{(x)}$ is parallel transported, $\nabla^x \omega^{(x)} = 0 \Rightarrow \nabla \omega^{(x)} = 0$.

$\nabla$ is metric-compatible $\implies g^{\mu\nu}(\omega^{(x)}, \omega^{(x)}) = \gamma^{\mu\nu}$ everywhere. (There are $\frac{1}{2}$ scales)

⇒ the $\omega^{(x)}$ form a basis everywhere (prove it!). $\gamma = \gamma_{\mu\nu} \omega^{(x)} \otimes \omega^{(x)}$.

$\nabla^x \omega^{(x)} = 0 \Rightarrow \nabla^x \omega^{(x)}|_p = 0 \Rightarrow \partial_x \omega^{(x)} - \omega^{(x)} = 0$ (in any coordinates).

Define the new coordinates $\xi^\lambda$ st. $\frac{\partial \xi^\lambda}{\partial x^\mu} = \omega^{(x)}$.

This does have a solution, because $\partial_x \omega^{(x)} = \partial_x \omega^{(x)}$.

$\Rightarrow \frac{\partial \xi^\lambda}{\partial x^\mu} = \frac{\partial \xi^\lambda}{\partial x^\alpha} dx^\alpha = \omega^{(x)} \frac{\partial \xi^\lambda}{\partial x^\mu} = \omega^{(x)} \frac{\partial \xi^\lambda}{\partial x^\mu}$ by definition.

$\Rightarrow g = \gamma_{\mu\nu} d\xi^\mu \otimes d\xi^\nu$. ⇒ we found a global flat coordinate system.
A useful relation. \(X, Y, Z\) vector fields.

\[
\nabla_x \nabla_y \nabla^2 = \nabla_x \nabla_y (Y^\beta \nabla_\beta X^\alpha) - \nabla_x Y^\beta (\nabla_\beta X^\alpha) = X^\alpha Y^\beta (\nabla_\alpha X^\beta - \nabla_\beta X^\alpha) \nabla_\beta X^\alpha
\]

\[
= X^\alpha Y^\beta \left( \nabla_\alpha X^\beta \nabla_\beta X^\alpha - \nabla_\beta X^\alpha \nabla_\beta X^\alpha \right) + [X, Y]^\beta \nabla_\beta X^\alpha
\]

\[
\text{Riemann}(-, X, Y, Z) = \nabla_x \nabla_y \nabla^2 - \nabla_x \nabla^2 - \nabla_x [X, Y] \nabla^2
\]

Geodesic deviation equation.

\[
\nabla_T (\nabla_S T) = \nabla_T (\nabla_S T) - \nabla_T \nabla_S T = \nabla_S \nabla_T T = \text{Riemann}(-, T, T, S)
\]

\[
\frac{D}{D^t} (\frac{DS}{D^t})^2 = R^\alpha_{\beta \delta \sigma} T^\beta T^\delta S^\sigma
\]

Interpretation.

The relative acceleration of neighboring geodesics is driven by tidal forces, embodied in the Riemann tensor.

Go to a locally inertial frame centered at a biducial trajectory (assumed timelike). In this frame, \(T^\nu = 0 \Rightarrow \) particle is stationary, with \(T^\nu = U^\nu = (1, 0, 0, 0)\).

\[
\frac{D X^\alpha}{D^r} = \frac{d X^\alpha}{d^r} \Rightarrow \frac{d S^\alpha}{d^2 r^2} = R^\alpha_{\beta \delta \sigma} S^\beta S^\delta = - R^\alpha_{\beta \delta \sigma} S^\beta S^\delta
\]

(Using antisymmetry of Riemann).
\[ \frac{\mathrm{d}^2 s^i}{\mathrm{d}t^2} = - R^i{}_{\rho\sigma\delta} s^\rho s^\sigma s^\delta \quad [g_{\mu\nu} = \eta_{\mu\nu} \text{ in locally inertial frame}] \]

s^i \text{ is the } \text{geodesic separation per unit } s \text{-coordinate.}

• **Newtonian limit.**

We already derived (Lecture 4) that in the Newtonian limit (slow motion, quasi-static, quasi-flat), geodesic equation is

\[ \frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = - \nabla_x \Phi, \quad \Phi = \frac{1}{2} (g_{oo} - \eta_{oo}) \]

In locally inertial coordinates, \( R^i{}_{\rho\sigma\delta} = \partial_{\rho} \partial_{\sigma} g_{\delta \gamma} - \partial_{\rho} \partial_{\gamma} g_{\delta \sigma} \)

\[ \Rightarrow R^i{}_{\rho\sigma\delta} = \partial_\rho \partial_\sigma g_{\delta \gamma} - \partial_\rho \partial_\gamma g_{\delta \sigma} \]

\[ \Rightarrow R^i{}_{\rho\sigma\delta} = \partial_\rho \partial_\sigma g_{\delta \gamma} - \partial_\rho \partial_\gamma g_{\delta \sigma} - \frac{1}{2} \partial_\gamma \partial_\rho g_{\delta \sigma} = \nabla_\rho \nabla_\sigma \Phi \]

\[ \Rightarrow \frac{\mathrm{d}^2 s^i}{\mathrm{d}t^2} = - \nabla_i \nabla_j \Phi \quad s^\delta. \]
Symmetries of the Riemann tensor. $\mathcal{R}_{\alpha\beta\gamma\delta} = \mathcal{R}_{\gamma\delta\alpha\beta}$

* $\nabla^\beta \mathcal{R}^{\alpha\beta\gamma\delta} = [\nabla_{\alpha}, \nabla_{\beta}] \mathcal{R}^{\gamma\delta} \Rightarrow$ antisymmetric in last 2 indices.

* From expression in local inertial frame, $\mathcal{R}_{\alpha\beta\gamma\delta} = -\mathcal{R}_{\beta\alpha\gamma\delta}$
  (tensornal equation, valid in one frame $\Rightarrow$ valid in all).

* $\mathcal{R}_{\alpha\beta\gamma\delta} = \mathcal{R}_{\gamma\delta\alpha\beta}$ (symmetric under exchange of 1st and 2nd pairs).

* $\mathcal{R}_{\alpha\beta\gamma\delta} + \mathcal{R}_{\kappa\sigma\delta\epsilon} + \mathcal{R}_{\kappa\delta\sigma\epsilon} = 0$ ($\Leftrightarrow \mathcal{R}_{\alpha\beta\gamma\delta} = 0$), when combined with other symmetries.

Number of independent components.

* Let's write $\mathcal{R}_{\alpha\beta\gamma\delta} = X_{\alpha\beta\gamma\delta} + \mathcal{R}[_{\alpha\beta\gamma\delta}]$.

For $\mathcal{R}_{\alpha\beta\gamma\delta}$, $\mathcal{R}[_{\alpha\beta\gamma\delta}]$ is antisymmetric under exchange of $\alpha \leftrightarrow \beta$ and $\sigma \leftrightarrow \delta$

symmetric under exchange of 1st and 2nd pair.

$\Rightarrow$ requiring Riemann to have these symmetries imply that $X_{\alpha\beta\gamma\delta}$ has them

$X$ has $N \frac{(N+1)}{2}$ components, with $N = \frac{n(n-1)}{2}$ number of "double" index.

* $\mathcal{R}[_{\alpha\beta\gamma\delta}] = \frac{1}{4} \left( \mathcal{R}_{\alpha\beta\gamma\delta} + \mathcal{R}_{\beta\alpha\gamma\delta} + \mathcal{R}_{\gamma\delta\alpha\beta} + \mathcal{R}_{\delta\gamma\alpha\beta} \right) = 0$

This represents $\frac{n(n-1)(n-2)(n-3)}{4!}$ independent constraints.

$\Rightarrow$ Number of independent components $= \frac{1}{2} \frac{n(n-1)}{2} \left( \frac{n(n-1)}{2} + 1 \right) - \frac{n(n-1)(n-2)(n-3)}{24}$

$= \frac{1}{12} n^4 (n^2 - 1)$

$= \text{number of linear combinations of 2nd derivatives of the metric that cannot be set to zero.}$