Master of science thesis in physics

The Membrane Vacuum State

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Abstract

In this thesis we make a review of membrane theory; presenting both the Lagrangian and the Hamiltonian formulation of the bosonic, as well as the supersymmetric, theory. The spectrum of the theories are derived and elaborated upon. The connection between membranes and Matrix theory is explicitly constructed, as is the case of dimensionally reduced super Yang-Mills theory.

We examine a two-dimensional supersymmetric $SU(2)$ invariant matrix model and prove that no normalizable ground state can exist for such a model. We then turn to a $SU(2) \times Spin(D-2)$ invariant matrix model corresponding to the regulated supermembrane propagating in D-dimensional spacetime, and formulate and prove a theorem stating that $D = 11$ is the only dimensionality for which an asymptotically normalizable ground state exists, the power law decay of which is also derived.

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Insane in the membrane
Insane in the brain!
- Cypress Hill, "Insane in the brain"

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1

Introduction

Membrane theory has a rather peculiar history and can trace its origins back to the very depths of the sixties, thus predating the emergence of its more illustrious and celebrated sibling, string theory. The birthplace of membrane theory, like many other grand ideas, was in the brilliant mind of Dirac [1]. In 1962, Dirac was pursuing an alternative model for the electron and put forth the hypothesis that it should correspond to a vibrating membrane. The resultant theory was plagued with many difficulties, however, and was soon abandoned to posterity. Some interest was later rekindled in the 1970's; along with the birth of string theory the strict adherence to a paradigm of a four-dimensional world containing zero-dimensional objects was seriously questioned (Dirac had been ahead of his time) and having thus in strings gone from zero to one dimension, the next step was conceptually easy. During this time the quantum mechanics of the membrane was analyzed, and the membrane itself was now utilized in various models for hadrons. Still plagued by many problems membrane theory was left in the cradle while string theory grew up fast and hit puberty, and eventually was elevated to the exalted rank of a candidate to the Theory of Everything. When the first "superstring revolution" hit the physics departments in 1984 and propelled string theory into respectable mainstream physics the membrane spectrum had been found continuous in the classical theory but discrete in the quantum case. This rare property of the Hamiltonian was investigated in [2] and is a very fortuitous trait as a continuous spectrum would spell disaster for a first quantized theory.

So far everything have concerned only the bosonic membrane, but if its aspirations are to describe Nature membrane theory must add fermions into the mix. The huge success in incorporating fermions with bosons in string theory via supersymmetry hinges on a crucial property called κ-symmetry. At first believed exclusive to strings, κ-symmetry was eventually generalized to membranes by Hughes, Liu and Polchinski [3] in 1986. As the newly christened supermembrane burst upon the scene a flurry of papers was published regarding the emergent theory. One important contribution was the realization that the type IIA superstring in ten dimensions could be obtained from the supermembrane in eleven dimensions by wrapping up one of the membrane's dimensions on a circle [4]. Another aspect which later turned out to be very intriguing and which plays a vital part in this thesis is the possibility to regularize the supermembrane [5] in terms of certain supersymmetric matrix models belonging to some finite group, e.g., $SU(N)$. The supermembrane is then recovered in the $N \to \infty$ limit. These matrix models were studied [6] some years prior to the discovery of this connection, and then in the context of dimensionally reduced supersymmetric Yang-Mills theory.

A problem now looming over the membrane community was the evidence pointing to a continuous spectrum for the quantum supermembrane. Until this was a proven fact, however, research continued unperturbed. The verdict came in a paper in 1989 [7], and the supermembrane was found guilty of continuity and subsequently condemned to the prison of bad ideas. Membrane theory thus lay dormant for some years until the second superstring revolution arrived in 1995 and revitalized the entire community. It now became apparent that the ten dimensions inherent in string theory was not enough to describe Nature. Furthermore, the five different string theories were unified and could trace themselves to a (M)other theory living in eleven dimensions, the very same dimensionality where the supermembrane assume its most appealing form. String theory was superseded by the newly baptized "M-theory" as the number one candidate for the final theory. Little was known about this mysterious theory other than that it had eleven-dimensional supergravity as a low-energy limit and that it tied together the various string theories with different dualities.

Many drastic breakthroughs were also made directly related to membranes. Townsend kick-started 1996 with a paper [8] suggesting that the continuous spectrum was not a failure of first quantization but instead implied a second quantized theory from the very beginning, thus turning the greatest weakness of the theory into a virtue. Slightly prior to this, Witten showed [9] a connection between D0-branes and matrix theory and thus tying them together with supermembranes. Based on this work Banks, Fischler, Shenker and Susskind made a bold conjecture claiming that these super matrix models in the large N limit *exactly* described M-theory in the infinite momentum frame. As these matrix models were inexorably linked to supermembranes interest exploded in matrix theory as well as membrane theory.

As membranes now became the focus of much research the old question of the existence and feasibility of obtaining its ground state arose again, this time with somewhat increased urgency as the stakes, and consequently the rewards, were considerably higher. Furthermore, matrix theory offered a new set of tools in attacking the problem. It turned out that the avenue of choice in confronting the membrane vacuum state is by trying to find the vacuum state of the corresponding $SU(N)$ matrix model. From 1997 and onwards work on this subject has been conducted with varying degrees of success. A large contribution to this field is due to the incessant efforts of Jens Hoppe and collaborators who have made promising advances mainly in the $SU(2)$ case, which is something we will study in detail in this thesis. While $N = 2$ is only a modest part of $N \to \infty$, its positive answer to the existence of a normalizable ground state can hopefully be generalized to arbitrary N.

We conclude this brief exposition of the history of membrane theory by stressing the fact that despite its bumpy ride between pre-eminence and obscurity membranes are now firmly established as an intricate part of M-theory. In light of this importance it is no surprise that our lack of an explicit membrane vacuum state or even proof that such a state exist is frustrating and simultaneously an incentive to continue the search for that state.

1.1 Outline of this thesis

This thesis is to a large part a review of membrane theory, although its content is strongly tilted towards tools needed to analyze the supermembrane vacuum state from the vantage point of matrix theory.

We begin the thesis with a treatment of the basic properties of the membrane. Before confronting the full-fledged supermembrane we carefully work through the less intimidating bosonic membrane, acquiring much needed tools and formalism along the way. Specifically, we will analyze the membrane in both Lagrangian and Hamiltonian formalism, discuss ways of quantization and take the membrane to the lightcone gauge, and discover an important residual symmetry, area preserving diffeomorphisms. We then continue with a very condensed introduction of supersymmetry, giving only the basics needed and swiftly moving on to discussing the viable choices of implementing supersymmetry into membrane theory. The stage is thus set for the entrance of the supermembrane, but instead we analyze what restrictions supersymmetry place on our theory. Strings and membranes are but two cases of the more general p-branes, extended objects of p dimensions. As we will show in the section entitled "the brane scan" supersymmetry gives us a very definite answer as to which values of p are allowed and in what dimensions of spacetime they can live in. Having done this we examine the action of the super p-brane and its attendant symmetries, an easy feat when we have the bosonic case still fresh in our minds.

The next chapter continues the treatment of supermembranes. We start with the Hamiltonian perspective and note the conceptually important area preserving diffeomorphisms (APD) and its implications on the dynamics of the membrane. The APD algebra is investigated further as we start building the bridge between supermembranes and matrix theory. We thus perform the regularization of the membrane. In the following section we treat another route to matrix theory, namely the dimensional reduction of super Yang-Mills theory. We then move on to analyze the spectrum of both the bosonic and supersymmetric membrane, in the bosonic case giving a full proof of the continuous spectrum for the classical theory and the discrete nature of the quantum case. For the supermembrane we illustrate the continuous spectrum by using a toy model.

The remaining part of chapter 3 takes on a slightly different flavor as we present a brief overview of what has become known as M-theory. The objective is in part to see the supermembrane in its natural surroundings and in part to give the author the opportunity to examine issues not directly connected with the subject matter of this thesis. Briefly, we touch upon subjects like supergravity, strings and duality. A slightly more in-depth discussion of the very important BFSS conjecture concludes the chapter.

The last chapter carries the same title as the thesis and is where we go into greater detail, narrowing our scope to the conjectured membrane vacuum state. After a short chronological overview of the research that has been done on the subject, a toy model ground state is investigated using a method similar to the one used in the subsequent theorem regarding the ground state of the $SU(2)$ invariant matrix model. We then present in detail the model and method used for analyzing this ground state, formulate the theorem [10] and examine the proof. The chapter is concluded by some remarks regarding a novel computational method that recently have been applied to similar problems.

In the three appendices we go through the notation and conventions used in the thesis, perform the proof of the κ-symmetry of the supermembrane, and finally present some calculations related to the main ground state theorem.

1.2 A remark

While I hold no illusion as to whether this thesis will ever be read beyond the acknowledgments, or perhaps this introduction, by anyone except my supervisor (who gets paid to do it), I feel compelled to be at least somewhat considerate of the readability of the text. Most scientific texts are by their very nature dull and austere, and this thesis is no exception. However, austerity can be a positive characteristic as it focuses on the only things of importance, without sugar-coating it. Moderation is as usual the key, as some authors take this scheme too far and condense their material into beautifully aesthetic pieces of writing, completely unreadable to a novice of the field.

While on the subject of aesthetics I would like to comment on chapter 4 and its complete lack of aesthetic expressions. Though some are just unattractive, the vast majority are plain hideous. The reasons for this is my strict adherence to keeping the notation of the original papers, which in this case spawned said obscenities. My only justification would be to remark that in an attempt to make the expressions more pleasing to the eye the clarity of the exposition might be lost; an unfair trade to be sure.

Despite the above, I have imagined a collection of enthusiastic and attentive readers and done my best to present the material in as a pedagogical way as possible, striving to heed the words of a man much wiser that myself: "Beware the lollipop of mediocrity. Lick once and you suck forever."

2

The Supermembrane

In this initial chapter we review the basic properties of the supermembrane. Emphasis is put on aspects relevant to the search for a membrane vacuum state. Although the majority of the material in this chapter applies to extended objects of higher (and lower) dimension than two, the 2-brane, or membrane, is ultimately the most interesting case for the purpose of this thesis and thus the one we will treat more thoroughly. The first part of this chapter concerns the bosonic theory and deals with membranes exclusively, while the later part of the chapter incorporates supersymmetry and treat the more general case of p-branes.

More in-depth treatments of the supermembrane abound; good ones include [11, 12, 13].

2.1 The bosonic membrane

In this section an introduction to the bosonic membrane is made. An understanding of bosonic membranes will be of great help when we want to treat the more involved supermembrane.

2.1.1 The membrane action

In the most general case we have a p-dimensional extended object, a p-brane, propagating in a curved target space of D dimensions. We will, however, restrict ourselves to the 2-brane in 11-dimensional spacetime. The reason for this will become clear when we discuss the supermembrane. For simplicity we also choose spacetime to be flat with Minkowski metric $\eta_{\mu\nu}$. In analogy with a particle or string sweeping out a worldline or worldsheet respectively, the time evolution of a membrane will create

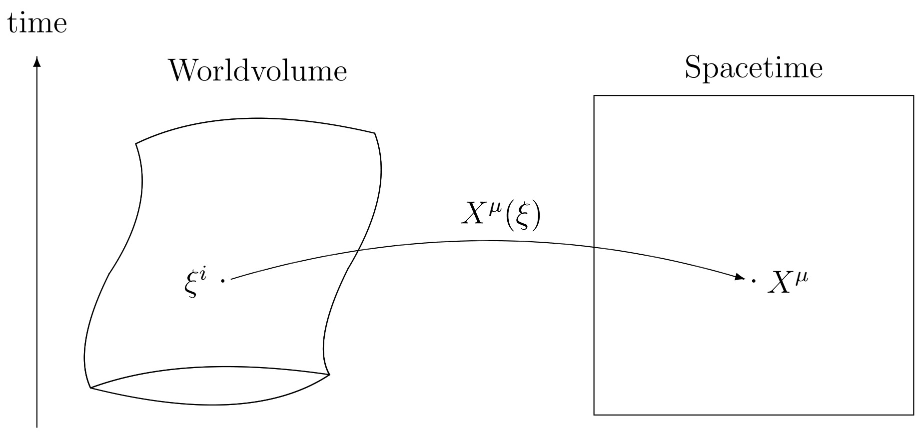

Figure 2.1: The mapping from worldvolume to spacetime.

a worldvolume. We choose the worldvolume coordinates to be

$$\xi^i = (\xi^0, \xi^1, \xi^2) = (\tau, \sigma^1, \sigma^2) \tag{2.1}$$

Furthermore we make a mapping from the worldvolume to the target space,

$$\xi \rightarrow X^\mu(\xi), \tag{2.2}$$

with the spacetime index μ going over $0, 1, \ldots, 10$. Next we want to construct the action of the membrane. To accomplish this we use the Nambu-Goto action principle, which equates the action with the worldvolume. We thus proceed by constructing a volume element from the induced metric on the worldvolume by pulling back the target space metric:

$$g_{ij}(X) = \partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu} \equiv E_i^\mu E_j^\nu \eta_{\mu\nu}, \tag{2.3}$$

where the E_i^μ is the dreibein to the worldvolume. We can now form a volume element and write down the action

$$S = -\int d^3\xi \sqrt{-g(X)}, \tag{2.4}$$

with g being the determinant of g_{ij}. This action was introduced by Dirac [1] when working on the hypothesis that electrons could be modelled as vibrating membranes. Nambu and Goto later used the above action for the string case. For the sake of simplicity the membrane tension T has been set to unity. T is a constant rendering the action dimensionless and can be brought back by simple dimensional analysis. A classically equivalent action,

$$S = -\int d^3\xi \sqrt{-g} g^{ij} E_i^{\mu} E_{j\nu} + \frac{1}{2}\sqrt{-g}, \tag{2.5}$$

was introduced by Howe and Tucker, and is commonly known as the Polyakov action. In this action g_{ij} is treated as an auxiliary field. However, if we vary the action with respect to g_{ij} we find that g_{ij} is just the induced metric. Substituting this into the Polyakov action we recover the Nambu-Goto action, thus the classical equivalence. As expected then, varying either action with respect to X^{μ} yields the same equations of motion,

$$\partial_i(\sqrt{-g} g^{ij} E_j^{\mu}) = 0. \tag{2.6}$$

The bosonic membrane exhibit a global symmetry determined by the target space geometry. In the case of $D = 11$ Minkowski space it is simply an invariance under eleven dimensional Poincaré transformations

$$\delta X^{\mu} = a^{\mu} + \omega^{\mu}{}_{\nu} X^{\nu}. \tag{2.7}$$

Local symmetry comes in the guise of worldvolume reparametrization invariance

$$\xi^i \to \xi^{i\prime}(\xi^0, \xi^1, \xi^2). \tag{2.8}$$

2.1.2 The Hamiltonian formulation

To leave the lagrangian formalism in favor of the hamiltonian variety we make use of Dirac's method for constrained hamiltonian systems [14]. We begin by forming the conjugate momenta

$$P_{\mu} = \frac{\delta \mathcal{L}}{\delta(\partial_0 X^{\mu})}, \tag{2.9}$$

where the lagrangian is of the Nambu-Goto type. If we introduce the following shorthand notation

$$\dot{X}^{\mu} = \frac{\partial X^{\mu}}{\partial \tau} \qquad X'^{\mu} = \frac{\partial X^{\mu}}{\partial \sigma^1} \qquad \bar{X}^{\mu} = \frac{\partial X^{\mu}}{\partial \sigma^2}, \tag{2.10}$$

the conjugate momenta takes the form

$$\begin{aligned} P_{\mu} \;&=\; \mathcal{L}^{-1}\Big(\dot{X}_{\mu}((X'\bar{X})^2 - X'^2\bar{X}^2) + X'_{\mu}((\dot{X}X')\bar{X}^2 \\ &\quad - (\dot{X}\bar{X})(X'\bar{X}))\bar{X}_{\mu}((\dot{X}\bar{X})X'^2 - (\dot{X}X')(X'\bar{X}))\Big). \end{aligned} \tag{2.11}$$

From the above we can then obtain a set of primary constraints for the membrane,

$$\phi_1 \equiv P \cdot X' \approx 0 \quad (2.12)$$
$$\phi_2 \equiv P \cdot \bar{X} \approx 0 \quad (2.13)$$
$$\phi_3 \equiv P^2 - (X'\bar{X})^2 X'^2 \bar{X}^2 \approx 0, \quad (2.14)$$

where ≈ 0 means "weakly zero", i.e., the constraints may have nonzero Poisson (or Dirac) brackets with the phase-space variables.

Without any primary constraints the Hamiltonian would simply be given by

$$H_0 = \int d\sigma^1 d\sigma^2 (\dot{X}^\mu P_\mu - \mathcal{L}). \quad (2.15)$$

As primary constraints are present, however, we can add arbitrary linear combinations of the constraints to H_0 without effecting the dynamics of the membrane. Furthermore, as H_0 is easily found to be zero the Hamiltonian is just constructed from the primary constraints alone,

$$H = \iint d\sigma^1 d\sigma^2 \left(\lambda(\tau,\sigma^1,\sigma^2)\phi_1 + \mu(\tau,\sigma^1,\sigma^2)\phi_2 + \nu(\tau,\sigma^1,\sigma^2)\phi_3\right), \quad (2.16)$$

with λ, μ, ν arbitrary functions. These functions in turn are set when we decide on a particular gauge. To move on to the equation of motion we first need to introduce the functional Poisson bracket,

$$\{g, H\} = \iint d\sigma^1 d\sigma^2 \left[\frac{\delta g}{\delta X(\tau,\sigma^1,\sigma^2)} \cdot \frac{\delta H}{\delta P(\tau,\sigma^1,\sigma^2)} - \frac{\delta g}{\delta P(\tau,\sigma^1,\sigma^2)} \cdot \frac{\delta H}{\delta X(\tau,\sigma^1,\sigma^2)}\right]. \quad (2.17)$$

Hamilton's equation of motion for a general dynamical functional of the form $g[P_\mu, X^\mu, \tau]$ is then

$$\dot{g} = \frac{dg}{dt} = \{g, H\} + \frac{\partial g}{\partial t}. \quad (2.18)$$

Now it is important to note that the above constraints can be used after the bracket operation has been performed, thus forcing λ, μ, ν to act as constants in the Poisson brackets. To see that the primary constraints are time-independent we must show that

$$\dot{\phi}_i = \{\phi_i, H\} \quad (2.19)$$

holds true on the subspace of the phase space where (2.12)-(2.14) are valid. In our case, where H_0 is identically zero, we only have to show that

$$\{\phi_i, \phi_j\} = C_{ijk}\phi_k, \quad (2.20)$$

where C_{ijk} are arbitrary functions of X^μ and P_μ. Calculating C_{ijk}, while not especially complicated, are painfully time-consuming and hence left as an exercise to the reader. In any case, a calculation will show that C_{ijk} are indeed functions of X^μ and P_μ only [15], i.e. the primary constraints are conserved. The dynamics of the membrane is then fully specified by the constraints (2.12)-(2.14) and the equation of motion (2.18) together with the full Hamiltonian

$$H = \iint d\sigma^1 d\sigma^2 \left(\lambda(X' \cdot P) + \mu(\bar{X} \cdot P) + \nu(P^2 - (X' \cdot \bar{X})^2 + X'^2 \bar{X}^2)\right). \tag{2.21}$$

The next logical step in our treatment of the membrane would be to proceed to a quantum theory. We are then presented with two choices. We can use covariant quantization to turn X^μ and P^μ into operators satisfying the canonical commutation relations and turn the constraints along with the Hamiltonian into operator expressions. The other choice, which we will treat later, entails reducing the degrees of freedom into an independent set of variables and then quantizing these variables. A virtue of the latter method is the lack of constraints in the quantum theory, while the downside is the loss of explicit covariance.

2.1.3 Covariant quantization

We begin by replacing the canonical variables with operators

$$X^\mu \to \hat{X}^\mu, \qquad P^\mu \to \hat{P}^\mu. \tag{2.22}$$

These operators are then required to satisfy the canonical commutation relations:

$$[\hat{X}^\mu(\tau,\sigma^1,\sigma^2), \hat{P}^\nu(\tau,\sigma'^1,\sigma'^2)] = 4\pi^2 i\hbar g^{\mu\nu}\delta(\sigma^1-\sigma'^1)\delta(\sigma^2-\sigma'^2) \tag{2.23}$$

$$[\hat{X}^\mu(\tau,\sigma^1,\sigma^2), \hat{X}^\nu(\tau,\sigma'^1,\sigma'^2)] = 0 \tag{2.24}$$

$$[\hat{P}^\mu(\tau,\sigma^1,\sigma^2), \hat{P}^\nu(\tau,\sigma'^1,\sigma'^2)] = 0. \tag{2.25}$$

To derive (2.23) and also to get rid of ordering ambiguities we have assumed H, X^μ and P^μ to be Hermitian. To proceed from (2.21) we make a gauge choice which will make the equation of motion for X^μ become a linear second order differential equation. To accomplish this we set the multipliers to

$$\lambda = 0, \qquad \mu = 0, \qquad \nu = \frac{1}{2}. \tag{2.26}$$

We now get the quantum Hamiltonian

$$H = \frac{1}{2}\iint d\sigma^1 d\sigma^2 \left(P^2 - (X'^2\bar{X}^2 + X' \cdot \bar{X})^2\right), \tag{2.27}$$

and the constraints (2.12)-(2.14) in operator form:

$$\begin{aligned}\hat{\phi}_1 &= X' \cdot P + P \cdot X' && (2.28)\\ \hat{\phi}_2 &= \bar{X} \cdot P + P \cdot \bar{X} && (2.29)\\ \hat{\phi}_3 &= P^2 - X'^2 \bar{X}^2 + (X' \cdot \bar{X})^2. && (2.30)\end{aligned}$$

These, in turn, are implemented by requiring

$$\hat{\phi}_i \, |P\rangle = 0, \tag{2.31}$$

for all physical states $|P\rangle$ (in the Heisenberg picture). However, actually putting the covariant quantization to some use is difficult. As in string theory ghosts would likely appear. In the string case the removal of the ghosts are made easy by the possibility to express the Hamiltonian in terms of creation and annihilation operators. The lack of creation and annihilation operators in the membrane case, however, would make said ghosts very troublesome to exorcise. We therefore drop the discussion of covariant quantization without further ado.

2.1.4 The lightcone gauge

To learn more about the membrane we need to choose a particular gauge in which to analyze the inherent physics of the membrane. As in string theory, the so-called lightcone gauge proves to be advantageous. To enter the lightcone gauge we first introduce the lightcone coordinates

$$X^{\pm} = \frac{1}{\sqrt{2}}(X^{10} \pm X^0) \tag{2.32}$$

and denote the transverse coordinates by

$$\vec{X}(\xi) = X^a(\xi), \qquad a = 1, \ldots, 9 \tag{2.33}$$

thus reducing the number of coordinates from 11 to 9. We then make the gauge choice

$$X^+(\xi) = X^+(0) + \tau, \tag{2.34}$$

hence implying

$$\partial_i X^+ = \delta_{i0}. \tag{2.35}$$

We can now write down the induced metric in the lightcone gauge:

$$\begin{aligned} g_{rs} &= \partial_r \vec{X} \cdot \partial_s \vec{X} \equiv \bar{g}_{rs}, \\ g_{0r} &= \partial_r X^- + \dot{\vec{X}} \cdot \partial_r \vec{X} \equiv u_r, \\ g_{00} &= 2\dot{X}^- + \dot{\vec{X}}^2. \end{aligned}$$

The determinant becomes

$$g = -\Delta \bar{g}, \tag{2.36}$$

where

$$\bar{g} \equiv det\bar{g}_{rs}, \qquad \bar{g}^{rs}\bar{g}_{st} = \delta^r_t, \qquad \Delta = -g_{00} + u_r\bar{g}^{rs}u_s. \tag{2.37}$$

By virtue of the above, the Lagrangian takes the simple form

$$\mathcal{L} = -\sqrt{\bar{g}\Delta}. \tag{2.38}$$

The Hamiltonian formulation is, however, of greater interest to us. We thus form the canonical momenta $\vec{P}$ and P^+ conjugate to $\vec{X}$ and X^-, respectively:

$$\vec{P} = \frac{\partial \mathcal{L}}{\partial \dot{\vec{X}}} = \sqrt{\frac{\bar{g}}{\Delta}}\left(\dot{\vec{X}} - u_r\bar{g}^{rs}\partial_s\vec{X}\right), \tag{2.39}$$

$$P^+ = \frac{\partial \mathcal{L}}{\partial \dot{X}^-} = \sqrt{\frac{\bar{g}}{\Delta}}. \tag{2.40}$$

The Hamiltonian density is

$$\mathcal{H} = \vec{P}\cdot\dot{\vec{X}} + P^+\dot{X}^- - \mathcal{L} = \frac{\vec{P}^2 + \bar{g}}{2P^+}. \tag{2.41}$$

With the primary constraint

$$\phi_r = \vec{P}\cdot\partial_r\vec{X} + P^+\partial_r X^- \approx 0 \tag{2.42}$$

and Lagrange multiplier c^r we construct the total Hamiltonian [14]

$$H_{total} = \int d^2\sigma\{\mathcal{H} + c^r\phi_r\}, \tag{2.43}$$

which has no secondary constraints. The gauge condition (2.34) has a residual invariance under the spatial diffeomorphism

$$\sigma^r \to \sigma^r + \xi^r(\tau,\sigma), \tag{2.44}$$

which will transform u^r as follows:

$$u^r \to u^r - \partial_0\xi^r + \partial_s\xi^r u^s - \xi^s\partial_s u^r. \tag{2.45}$$

This will allow us to impose the gauge condition

$$u^r = 0, \tag{2.46}$$

since "$\partial_0 \xi^r$" is independent of u^r and can be chosen to cancel the other terms. From this it follows that the Hamilton equations corresponding to H_{total} imply that $c^r = 0$ and moreover that

$$\partial_0 P^+ = 0. \tag{2.47}$$

As $P^+(\sigma)$ transforms as a density under diffeomorphisms we can rewrite it as a constant times a density, i.e.,

$$P^+(\sigma) = P_0^+ \sqrt{w(\sigma)}, \tag{2.48}$$

where we will normalize the function $\sqrt{w(\sigma)}$ according to

$$\int d^2\sigma \sqrt{w(\sigma)} = 1. \tag{2.49}$$

The function $w_{rs}(\sigma)$ can be interpreted as a 2-by-2 spatial metric on the membrane itself, and $w(\sigma)$ as the metric determinant. Except for being non-singular the metric is arbitrary. However, it is important to note that no physical quantity can be allowed to depend on our choice of metric. This independence is a consequence of the invariance of the theory under area preserving diffeomorphisms, together with the fact that, except for the Lorentz boost generators, the metric $w_{rs}(\sigma)$ only appears in various physical quantities in the guise of the metric determinant $\sqrt{w(\sigma)}$. Furthermore, area preserving diffeomorphisms, which will be treated in the next section, leave by definition $\sqrt{w(\sigma)}$ invariant. From the above expressions we see that P_0^+ is the membrane momentum in the X^- direction, and

$$P_0^+ = \int d^2\sigma P^+. \tag{2.50}$$

The center of mass momenta is given by

$$\vec{P}_0 \equiv \int d^2\sigma \vec{P}(\sigma), \tag{2.51}$$

$$P_0^- \equiv \int d^2\sigma \mathcal{H}, \tag{2.52}$$

whereby the mass formula for the membrane becomes

$$\mathcal{M}^2 = -2P_0^+ P_0^- - \vec{P}_0^2 = \int d^2\sigma \left\{ \frac{[\vec{P}^2]' + \bar{g}}{\sqrt{w(\sigma)}} \right\}. \tag{2.53}$$

The relation between the Hamiltonian and mass being

$$H = \mathcal{M}^2 = T + V. \tag{2.54}$$

The meaning of the prime in equation (2.53) is the exclusion of the zero mode, $\vec{X}_0$, defined by

$$\vec{X}_0 \equiv \int d^2\sigma \sqrt{w(\sigma)}\vec{X}(\sigma). \tag{2.55}$$

The center of mass kinematics are determined by the theory of a free relativistic particle, while the membrane dynamics are governed by equation (2.53). An obvious observation regarding the mass formula is the lack of explicit dependence on the coordinate X^-. This coordinate is instead determined by the gauge condition $u^r = 0$, i.e.,

$$\partial_r X^- = -\dot{\vec{X}} \cdot \partial_r \vec{X}. \tag{2.56}$$

For X^- to be a globally defined function of σ^r

$$\oint (\partial_0 \vec{X} \cdot \partial_r \vec{X}) = 0 \tag{2.57}$$

must be fulfilled for any closed curve on the membrane. This locally amounts to the condition

$$\phi = \epsilon^{rs}(\partial_r \vec{P} \cdot \partial_s \vec{X}) \approx 0. \tag{2.58}$$

2.1.5 Area preserving diffeomorphisms

The gauge condition (2.58) used in the previous section leave a residual gauge symmetry of the lightcone Hamiltonian. This reparametrization invariance answers to the name of APD, area preserving diffeomorphisms.

Let us introduce a bracket operation on any two functions $A(\sigma)$ and $B(\sigma)$ in the shape of

$$\{A, B\}(\sigma) \equiv \frac{\epsilon^{rs}}{\sqrt{w(\sigma)}} \partial_r A(\sigma) \partial_s B(\sigma), \tag{2.59}$$

which is manifestly antisymmetric,

$$\{A, B\} = -\{B, A\}, \tag{2.60}$$

and satisfies the Jacobi identity

$$\{A, \{B, C\}\} + \{C, \{A, B\}\} + \{B, \{C, A\}\} = 0. \tag{2.61}$$

The bracket is thus a Lie bracket and together with the functions on the membrane forms an infinite dimensional Lie algebra. We can now rewrite the potential density $\bar{g}$ as

$$\bar{g} = \{X^a, X^b\}^2, \tag{2.62}$$

which turn the Hamiltonian (2.53) into

$$\mathcal{M}^2 = \frac{1}{2}\int d^2\sigma \left\{ \frac{[\vec{P}^2(\sigma)]'}{\sqrt{w(\sigma)}} + \sqrt{w(\sigma)}\{X^a, X^b\}^2 \right\}. \tag{2.63}$$

From this expression we can deduce that $\{X^a, X^b\}^2$ is a measure of the potential energy of the membrane. From the fact that

$$\{X^a, X^b\} = \frac{\epsilon^{rs}}{\sqrt{w(\sigma)}}\partial_r X^a \partial_s X^b \tag{2.64}$$

is just the area element of the membrane pulled back into spacetime we conclude that a membrane can change its shape and form, as long as the area remains constant, without any change in its potential energy. This is of course the meaning of area preserving diffeomorphisms and correspond to the transformation

$$\sigma^r \to \sigma^r + \xi^r(\sigma) \tag{2.65}$$

with

$$\partial_r(\sqrt{w(\sigma)}\xi^r(\sigma)) = 0. \tag{2.66}$$

Locally, this amounts to

$$\xi^r(\sigma) = \frac{\epsilon^{rs}}{\sqrt{w(\sigma)}}\partial_s \xi(\sigma). \tag{2.67}$$

It is also worth noting that the variation of a function f under an infinitesimal area preserving reparametrization is

$$\delta f = -\xi^r \partial_r f = \{\xi, f\}. \tag{2.68}$$

Furthermore we can rewrite the constraint (2.58) in the form

$$\phi(\sigma) \equiv \{\vec{P}, \vec{X}\} \tag{2.69}$$

and verify that the mass $\mathcal{M}$ actually commutes with this constraint.

2.2 Supersymmetry

The field of supersymmetry has since its birth in the early 1970s grown to become one of the most expansive and fundamental theories within the domain of theoretical physics. It would be hubris to attempt to pen down a self-contained review of such an encompassing field, so excluding some brief introductory words this section will only present facts pertinent to membranes. Numerous good reviews of supersymmetry exists; if unable to get ones hands on the elusive [16] the more readily available [17] and [18] will suffice.

2.2.1 Basics

Supersymmetry is a proposed symmetry linking fermions and bosons together. In essence, every spin-$\frac{1}{2}$ particle would have a spin-1 sibling and vice versa. The properties of these twin particles would be the same as the original ones except for the spin and the name which, incidentally, is always more funny sounding for the supersymmetric twin (slepton, wino, higgsino etc.).

The only symmetries possible for the S-matrix (and hence also the Lagrangian) in particle physics was shown in a classical paper in 1967 by Coleman and Mandula [19] to be:

- Spacetime symmetries, Poincaré invariance, i.e., the semi-direct product of translations and Lorentz rotations.
- Internal global symmetries, related to conservation of quantum numbers like electric charge and isospin.
- Discrete symmetries, C(harge), P(arity) and T(ime).

In deriving the Coleman-Mandula theorem one of the assumptions made is that the S-matrix involves only commutators. By relaxing this constraint and allowing for anticommutating generators as well, room is made for supersymmetry; an extension of the spacetime symmetry mentioned above (resulting in a super-Poincaré algebra). Later, in 1975, it was proved [20] that supersymmetry is the *only* additional symmetry allowed under the aforementioned assumptions.

The operator Q that realizes supersymmetry is thus an anticommutating spinor, with

$$Q\,|\text{Boson}\rangle = |\text{Fermion}\rangle \quad \text{and} \quad Q\,|\text{Fermion}\rangle = |\text{Boson}\rangle\,. \tag{2.70}$$

Q and its hermitian conjugate $Q^\dagger$ is then by the extension of the Coleman-Mandula theorem forced to satisfy the algebra

$$\{Q, Q^\dagger\} = P^\mu \tag{2.71}$$
$$\{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0 \tag{2.72}$$
$$[P^\mu, Q] = [P^\mu, Q^\dagger] = 0, \tag{2.73}$$

where we have suppressed the spinor indices on Q and $Q^\dagger$, and where P^μ is the generator of translations.

Irreducible representations of the supersymmetry algebra called supermultiplets contain both bosonic and fermionic states (superpartners of each other). If $|\Omega\rangle$ and $|\Omega'\rangle$ belong to the same supermultiplet, then $|\Omega'\rangle$

is by definition proportional to some combination of Q and $Q^\dagger$ acting on the superpartner state $|\Omega\rangle$ (up to spacetime translations and rotations). Now, since the (mass)^2 operator, $-P^2$, commutes with Q, $Q^\dagger$ and all spacetime translation and rotation operators the particles contained in the same supermultiplet must necessarily have identical eigenvalues of $-P^2$ and hence equal mass. In addition to this, the operators Q and $Q^\dagger$ commute with the generators of gauge transformations. This implies the fact that superpartners share the same properties of color degrees of freedom, electric charge and weak isospin.

To make the transition from bosonic membranes to supermembranes we have to introduce supersymmetry. We are then presented with two obvious choices; either we introduce supersymmetry locally on the membrane worldvolume (producing a so-called *spinning* membrane) or on the target space (yielding *superspace*). A third option would be to simultaneously make use of both these approaches, resulting in something called superembeddings [21]. This last option, though perhaps the most desirable, is plagued with many difficulties and while research in this field is still being conducted we will not touch on the subject any further.

2.2.2 Worldvolume supersymmetry

Introducing supersymmetry on the worldvolume of a p-brane would produce (p+1)-dimensional "matter" supermultiplets (X^μ, χ^μ), with $\mu = 0, 1, \ldots, D$ and χ a worldvolume spinor. In the string case, $p = 1$, this yields the spinning string of Ramond, and Neveu and Schwarz. After some GSO projections the obtained spectrum is exactly that of a spacetime supersymmetric theory. Furthermore, the resulting action is equivalent to the normal Green-Schwarz superstring action. An attempt to construct a spinning membrane was made in [22]. However, the above procedure led to the inclusion of an Einstein-Hilbert term, $\sqrt{-g}R$, when trying to supersymmetrize the cosmological term $\sqrt{-g}$ in the bosonic action. These problems later grew into a no-go theorem for spinning membranes [23].

2.2.3 Superspace

The obvious alternative to the spinning membrane is to introduce supersymmetry on the background space. This is accomplished simply by adding a number of anticommutating fermionic coordinates, $\theta^\alpha(\xi)$, to the bosonic ones, $X^\mu(\xi)$. Thus yielding the superspace coordinates

$$Z^M = (X^\mu, \theta^\alpha). \tag{2.74}$$

The fermionic coordinates, however, do not represent any position in spacetime *per se*, and is better viewed as "directions".

2.2.4 Experimental verification of supersymmetry

Supersymmetry in its unbroken form, with superparticles being degenerate in mass with the "normal" particles, is clearly impossible as such superparticles easily should have been found by now. Thus, if supersymmetry exists it does so in a broken form. Various calculations of the resulting superparticle masses (see e.g. [24] and [25]) place them in the TeV range. With the advent of the Large Hadron Collider at CERN in 2006 this energy range will soon come within experimental reach. If experimental evidence for supersymmetry were to be found it would be a tremendous victory (and relief) not only for the supersymmetry community but for string theorists in general as well. It would furthermore excite them to thenceforth "eat raspberry cake every day" [26].

2.3 Super p-branes

In this section we meld the two previous ones to produce super p-branes. We also present the brane scan, which will tell us the viable values of p. Furthermore we analyze the super p-brane action along with its symmetries.

2.3.1 The brane scan

For many students of string theory the first surprise to come to terms with is the leap from their childhood world of three dimensions to the mind-boggling world of string theory with a seemingly arbitrarily chosen number of dimensions. A simple but powerful way to bring some method to the madness is to make a so-called brane scan [27]. This will provide us with the allowed dimensions of p-branes for various dimensions of spacetime. It will also place an upper limit on the dimensionality of spacetime itself.

Consider a p-brane with worldvolume coordinates ξ^i $(i = 0, 1, \ldots, p)$ that moves through a D-dimensional spacetime. The p-brane traces a trajectory described by the functions $X^M(\xi)$, with $M = 0, 1, \ldots, D-1$. We then enter the static gauge by splitting the functions into

$$X^M(\xi) = (X^\mu(\xi), Y^m(\xi)), \tag{2.75}$$

where $\mu = 0, 1, \ldots, p$ and $m = p+1, \ldots, D-1$. Next we put

$$X^\mu(\xi) = \xi^\mu, \tag{2.76}$$

which results in the physical degrees of freedom being given by the $(D-p-1)Y^m(\xi)$. Consequently, the number of bosonic degrees of freedom on-shell is

$$N_B = D - p - 1. \tag{2.77}$$

To get a super p-brane we enter superspace by adding the fermionic coordinates $\theta^\alpha(\xi)$ to the bosonic $X^\mu(\xi)$. The number of fermionic degrees of freedom is naively the number of real components, M, of the minimal spinor times the number of supersymmetries, N. However, this product is then halved by κ-symmetry[1]. Then by going on-shell only half of the spinor components will be identified as coordinates and the other half as momenta. It is important to note that this reasoning only applies when $p > 1$. The string case is slightly more complicated as we can treat left and right moving modes separately. If we first consider $p > 1$ the number of fermionic degrees of freedom on-shell is

$$N_F = \frac{1}{4}MN. \tag{2.78}$$

To implement supersymmetry on the membrane we must enforce $N_B = N_F$, i.e.,

$$D - p - 1 = \frac{1}{4}MN. \tag{2.79}$$

We can then easily deduce the dimensionality of the spacetime allowed for a certain dimension of the p-brane with the help of table 2.1, listing the number of supersymmetries and minimal spinor components for a given dimension. For the case of $p = 1$ we have two options; either we require matching of the number of right (or left) moving bosons and fermions, or the sum of both right and left. In the first case (2.79) is replaced by

$$D - 2 = \frac{1}{2}MN, \tag{2.80}$$

and has solutions for $D = 3, 4, 6$ and 10, all with $N = 1$. Furthermore these solutions actually describe the heterotic string. In the second case (2.79) remains valid and we get the same solutions for D as above, except that $N = 2$, and describe Type II superstrings.

A very fundamental conclusion to be drawn from the above kind of reasoning is that the maximum possible dimension allowed for spacetime is eleven. If $D \geq 12$ then $M \geq 64$, for which there are no solutions to (2.79). Likewise the upper limit on the dimensionality of the p-brane is five since (2.79) has no solution for $p \geq 6$. The results of the brane scan are summarized in figure 2.2. From the figure we can easily see that

[1] This important symmetry is examined more thoroughly in appendix B.

Dimension D	Minimal spinor M	Supersymmetries N
11	32	1
10	16	1,2
9	16	1,2
8	16	1,2
7	16	1,2
6	8	1,2,3,4
5	8	1,2,3,4
4	4	1,...,8
3	2	1,...,16
2	1	1,...,32

Table 2.1: Number of minimal spinor components and supersymmetries for given spacetime dimensions.

the allowed dimensions are organized in four sequences. They are known as the real ($\mathcal{R}$), complex ($\mathcal{C}$), quaternionic ($\mathcal{H}$) and octonionic ($\mathcal{O}$) sequences, and are related to the composition-division algebras $\mathcal{R}, \mathcal{C}, \mathcal{H}, \mathcal{O}$.

Until now we have used only classical considerations to arrive at these restrictions of dimensionalities. Quantum mechanics can be expected to throw further restrictions into the mix. In string theory we know that the $D = 10$ string, i.e. the string that belongs to the octonionic sequence, is the only one free from quantum anomalies. In fact it can be shown for the p-brane that all the other sequences suffer from Lorentz anomalies in the lightcone gauge. We are thus tempted to conclude that the $p = 2$ brane in eleven dimensional spacetime is the only viable candidate for a fundamental super p-brane besides the $D = 10$ superstring. Later on we will also find a very close connection between the $D = 11$ supermembrane and $D = 11$ supergravity. For these reasons we will narrow our scope and focus, for the most part, on the $D = 11$ supermembrane.

2.3.2 The super p-brane action

Now we have the necessary tools to construct the supersymmetric generalization of the the bosonic membrane in section 2.1.1.

We start with a general p-brane in a D-dimensional flat superspace. The worldvolume is then parameterized by the local coordinates

$$\xi^i = (\tau, \sigma^r), \tag{2.81}$$

where $r, s, \ldots = 0, 1, \ldots, D-1$. As in the bosonic case we first construct

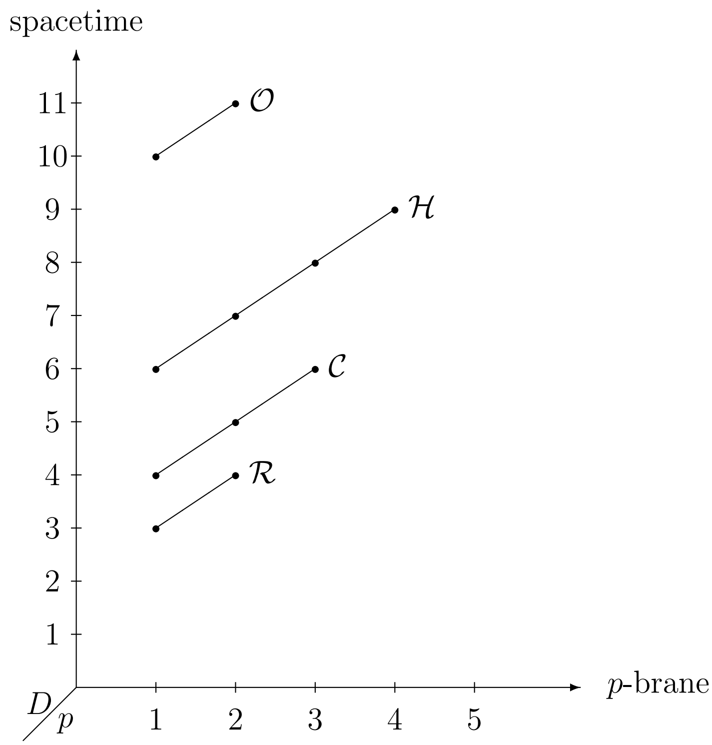

Figure 2.2: The brane scan.

the induced metric

$$g_{ij}(X,\theta) = E_i^\mu E_j^\nu \eta_{\mu\nu}. \tag{2.82}$$

In superspace we have a supervielbein of the form

$$E_i^\mu \equiv \partial_i X^\mu + \bar{\theta}\Gamma^\mu \partial_i \theta, \tag{2.83}$$

where we have introduced the anticommutating Γ-matrices of the Clifford algebra[2],

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}. \tag{2.84}$$

By treating g_{ij} as an independent variable on the worldvolume we get the Polyakov action for the super p-brane,

$$\mathcal{L} = -\frac{1}{2}\sqrt{-g}g^{ij}E_i^\mu E_{j\mu} + \frac{1}{2}(p-1)\sqrt{-g}, \tag{2.85}$$

where, as before, we have set the membrane tension to unity. In the same way as in the bosonic case we can go to the classically equivalent Nambu-Goto-like action by solving the equations of motion and substituting the

[2]Further information on Γ-matrices is given in appendix A.

on-shell metric. For the supermembrane in flat eleven-dimensional target space this action is

$$\mathcal{L} = -\sqrt{-g} - \epsilon^{ijk}\left(\frac{1}{2}\partial_i X^\mu \partial_j X^\nu + \frac{1}{2}\partial_i X^\mu \bar{\theta}\Gamma^\nu \partial_j \theta + \frac{1}{6}\bar{\theta}\Gamma^\mu \partial_i \theta \bar{\theta}\Gamma^\nu \partial_j \theta\right)\bar{\theta}\Gamma_{\mu\nu}\partial_k \theta. \quad (2.86)$$

From the Polyakov action we obtain the Euler-Lagrange equations of motion,

$$\partial_i(\sqrt{-g}g^{ij}E_j^\mu) = \epsilon^{ijk}E_j^\nu \partial_j \bar{\theta}\Gamma^\mu_\nu \partial_k \theta, \quad (2.87)$$

$$(1+\Gamma)g^{ij}\not{E}_i \partial_j \theta = 0, \quad (2.88)$$

where

$$\Gamma \equiv \frac{\epsilon^{ijk}}{6\sqrt{-g}}E_i^\mu E_j^\nu E_k^\rho \Gamma_{\mu\nu\rho} \quad (2.89)$$

and $\Gamma^2 = 1$ (for a proof, see appendix B), thus making $1 \pm \Gamma$ projection operators, eliminating half the spinor components.

The symmetries of the action is:

- Global super-Poincaré transformations,

$$\delta X^\mu = a^\mu + \omega^{\mu\nu}X_\nu - \bar{\epsilon}\Gamma^\mu \theta \quad (2.90)$$

$$\delta\theta = \frac{1}{4}\omega_{\mu\nu}\Gamma^{\mu\nu}\theta + \epsilon, \quad (2.91)$$

  where ϵ is a constant anticommuting spacetime spinor.

- Local gauge symmetry, in the form of worldvolume reparametrization invariance along a vector field ζ and a fermionic κ-symmetry,

$$\delta X^\mu = \zeta^i \partial_i X^\mu + \bar{\kappa}(1-\Gamma)\Gamma^\mu \theta \quad (2.92)$$

$$\delta\theta = \zeta^i \partial_i \theta + (1-\Gamma)\kappa, \quad (2.93)$$

with κ a 32-component Majorana spinor. Via Noether's theorem we can obtain the supercharges (supersymmetry generators)

$$Q = \int d^2\sigma J^0, \quad (2.94)$$

with the conserved supercurrent being

$$J^i = -2\sqrt{-g}g^{ij}\not{E}_j\theta - \epsilon^{ijk}\left\{E_j^\mu E_k^\nu \Gamma_{\mu\nu}\theta + \frac{4}{3}[\Gamma^\nu\theta(\bar{\theta}\Gamma_{\mu\nu}\partial_j\theta) + \Gamma_{\mu\nu}\theta(\bar{\theta}\Gamma^\nu\partial_j\theta)](E_k^\mu - \frac{2}{5}\bar{\theta}\Gamma^{\mu\nu}\partial_k\theta)\right\}. \quad (2.95)$$

We conclude this chapter by presenting the supermembrane action in a curved background space. The action was proposed by Bergshoeff, Sezgin and Townsend in 1987 [28] and investigated extensively later that year [29]. The action presented below differ, however, from their action by a factor 1/3! in the last term due to slightly different conventions. The action is,

$$S = \int d^3\xi \left(-\frac{1}{2}\sqrt{-g}g^{ij}\Pi_i^a\Pi_j^b\eta_{ab} + \frac{1}{2}\sqrt{-g} - \frac{1}{3!}\epsilon^{ijk}\Pi_i^A\Pi_j^B\Pi_k^C B_{CBA} \right), \tag{2.96}$$

where indices A, B, C are flat super indices and a, b are flat vector indices $(A = a, \alpha)$. The pullback is defined as,

$$\Pi_i^A = (\partial_i Z^M)E_M{}^A, \tag{2.97}$$

with $E_M{}^A$ the supervielbein and

$$E^A = dZ^M E_M{}^A. \tag{2.98}$$

The 3-form B is the potential to the 4-form H,

$$H = dB, \tag{2.99}$$

with B being defined as

$$B = \frac{1}{3!}E^A E^B E^C B_{CBA}. \tag{2.100}$$

The action has two local gauge invariances; local fermionic κ-symmetry (investigated in detail in appendix B), and $d = 3$ reparametrization invariance,

$$\delta Z^M = \eta^i(\xi)\partial_i Z^M \tag{2.101}$$

$$\delta g_{ij} = \eta^k\partial_k g_{ij} + 2\partial_{(i}\eta^k g_{j)k}. \tag{2.102}$$

3

The Supermembrane: Spectrum and M-Theory

This chapter has three parts. We begin with a treatment of the link between matrix theory and supermembranes, then move on to investigate the membrane spectrum. The last part is a brief overview of what has become known as M-theory.

3.1 Supermembranes and matrix theory

In this section we will continue our treatment of the supermembrane and highlight its connection to matrix theory.

3.1.1 Lightcone gauge and Hamiltonian formalism

As before we enter the lightcone gauge by introducing the standard lightcone coordinates

$$X^{\pm} = \frac{1}{\sqrt{2}}(X^{10} \pm X^{0}), \tag{3.1}$$

and imposing the condition

$$X^{+}(\xi) = X^{+}(0) + \tau \Longleftrightarrow \partial_i X^{+} = \delta_{i0}. \tag{3.2}$$

Transverse coordinates are $\vec{X}(\xi) = X^a(\xi)$, with $a = 1, \ldots, 9$. In complete analogy for the gamma matrices, we define

$$\Gamma^{\pm} = \frac{1}{\sqrt{2}}(\Gamma^{10} \pm \Gamma^{0}). \tag{3.3}$$

The κ-symmetry is gauged fixed by imposing the gauge condition

$$\Gamma^+\theta = 0, \tag{3.4}$$

thus reducing the number of fermionic degrees of freedom from 32 to 16. After these substitutions the induced metric becomes

$$\begin{aligned} g_{rs} &= \partial_r\vec{X}\cdot\partial_s\vec{X} \equiv \bar{g}_{rs}, && (3.5)\\ g_{0r} &= \partial_r X^- + \partial_0\vec{X}\cdot\partial_r\vec{X} + \bar{\theta}\Gamma^-\partial_r\theta \equiv u_r, && (3.6)\\ g_{00} &= 2\partial_0 X^- + (\partial_0\vec{X})^2 + 2\bar{\theta}\Gamma^-\partial_0\theta. && (3.7) \end{aligned}$$

Furthermore, the metric determinant can be written as,

$$g \equiv -\Delta\bar{g}, \tag{3.8}$$

with, as before,

$$\bar{g} \equiv \det\bar{g}_{rs}, \qquad \bar{g}^{rs}\bar{g}_{st} = \delta^r_t, \qquad \Delta = -g_{00} + u_r\bar{g}^{rs}u_s. \tag{3.9}$$

The lightcone Lagrangian then becomes

$$\mathcal{L} = -\sqrt{\bar{g}\Delta} + \epsilon^{rs}\partial_r X^a\bar{\theta}\Gamma^-\Gamma_a\partial_s\theta. \tag{3.10}$$

To obtain the Hamiltonian density we first calculate the canonical momenta $\vec{P}$, P^+ and S (conjugate to $\vec{X}$, X^- and θ):

$$\begin{aligned} \vec{P} &= \frac{\partial\mathcal{L}}{\partial(\partial_0\vec{X})} = \sqrt{\frac{\bar{g}}{\Delta}}\left(\partial_0\vec{X} - u_r\bar{g}^{rs}\partial_s\vec{X}\right), && (3.11)\\ P^+ &= \frac{\partial\mathcal{L}}{\partial(\partial_0 X^-)} = \sqrt{\frac{\bar{g}}{\Delta}} && (3.12)\\ S &= \frac{\partial\mathcal{L}}{\partial(\partial_0\bar{\theta})} = -\sqrt{\frac{\bar{g}}{\Delta}}\Gamma^-\theta. && (3.13) \end{aligned}$$

The Hamiltonian density is then

$$\begin{aligned} \mathcal{H} &= \vec{P}\cdot\partial_0\vec{X} + P^+\partial_0 X^- + \bar{S}\partial_0\theta - \mathcal{L} && (3.14)\\ &= \frac{\vec{P}^2 + \bar{g}}{2P^+} - \epsilon^{rs}\partial_r X^a\bar{\theta}\Gamma^-\Gamma_a\partial_s\theta, && (3.15) \end{aligned}$$

and the Hamiltonian itself being the integral of the above density over the membrane, i.e.,

$$H = \int_{\mathcal{M}} d^2\sigma\,\mathcal{H}(\sigma). \tag{3.16}$$

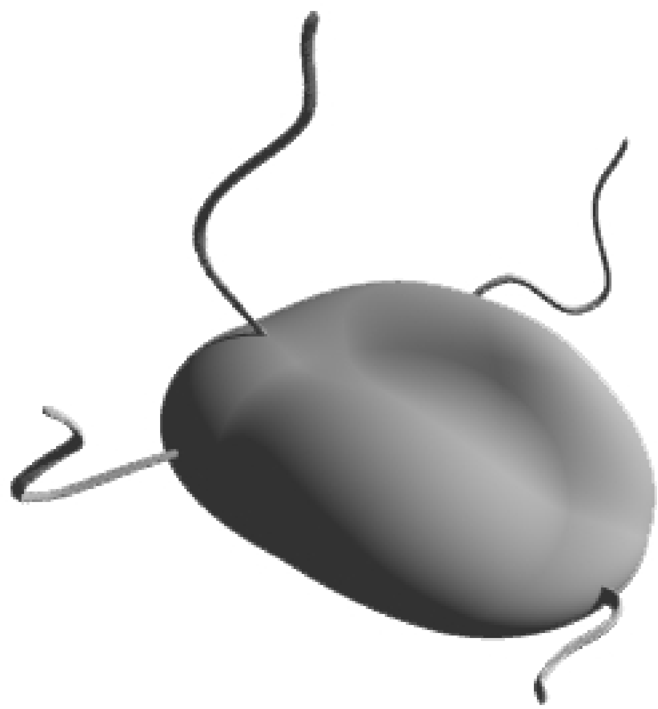

Figure 3.1: A membrane with protruding tubes.

The bosonic part of (3.14) was first discovered by Goldstone [30] with the fermionic part incorporated in [31]. The two primary constraints are

$$\phi_r = \vec{P}\cdot\partial_r\vec{X} + P^+\partial_r X^- + \bar{S}\partial_r\theta \approx 0 \tag{3.17}$$
$$\chi = S + P^+\Gamma^-\theta \approx 0. \tag{3.18}$$

In complete analogy with the reasoning in section 2.1.4 we can impose the gauge condition

$$u^r = 0, \tag{3.19}$$

and introduce the normalized spatial metric $w(\sigma)$, and then produce the membrane momenta

$$P_0^+ = \int d^2\sigma P^+, \tag{3.20}$$

$$\vec{P}_0 = \int d^2\sigma \vec{P}, \tag{3.21}$$

$$P_0^- = \int d^2\sigma \mathcal{H}. \tag{3.22}$$

The membrane mass then becomes

$$\mathcal{M}^2 = \int d^2\sigma \left\{ \frac{[\vec{P}^2]' + \bar{g}}{\sqrt{w(\sigma)}} - 2P_0^+ \epsilon^{rs} \partial_r X^a \bar{\theta} \Gamma^- \Gamma_a \partial_s \theta \right\}, \tag{3.23}$$

the prime again indicating the exclusion of the zero modes

$$\vec{X}_0 = \int d^2\sigma \sqrt{w(\sigma)} \vec{X}(\sigma) \tag{3.24}$$

$$\theta_0 = \int d^2\sigma \sqrt{w(\sigma)} \theta(\sigma). \tag{3.25}$$

Due to the fact that the bosonic part of (3.23) is

$$\mathcal{M}^2 = H = T + V \tag{3.26}$$

we obtain the potential energy

$$V = \int d^2\sigma \bar{g} = \int d^2\sigma \det_{r,s}(\partial_r \vec{X} \cdot \partial_s \vec{X}) = \int d^2\sigma (\epsilon^{rs} \partial_r X^a \partial_s X^b)^2. \tag{3.27}$$

From this expression we deduce that the potential energy will vanish where the membrane is infinitely thin (i.e., where the $\vec{X}$'s depend on one linear combination of the σ's only). Hence the membrane can sprout stringlike spikes without any cost in energy. Although we could have surmised this by way of area preserving diffeomorphisms (as the strings have zero area), there is a deeper meaning; the spikes do not necessarily need to have a "stringy" end. A membrane could, e.g., squeeze its midsection into a string (not a string *per se*, but an infinitesimally thin tube), effectively becoming two membranes connected with a string. As pointed out earlier this string would not carry any energy and the case where two membranes are connected with a string would actually be physically indistinguishable from the case without the string connection. This is a remarkable feature of membrane theory: if membranes can join and disjoin freely any concept of a "membrane number" (conserved or not) becomes irrelevant.

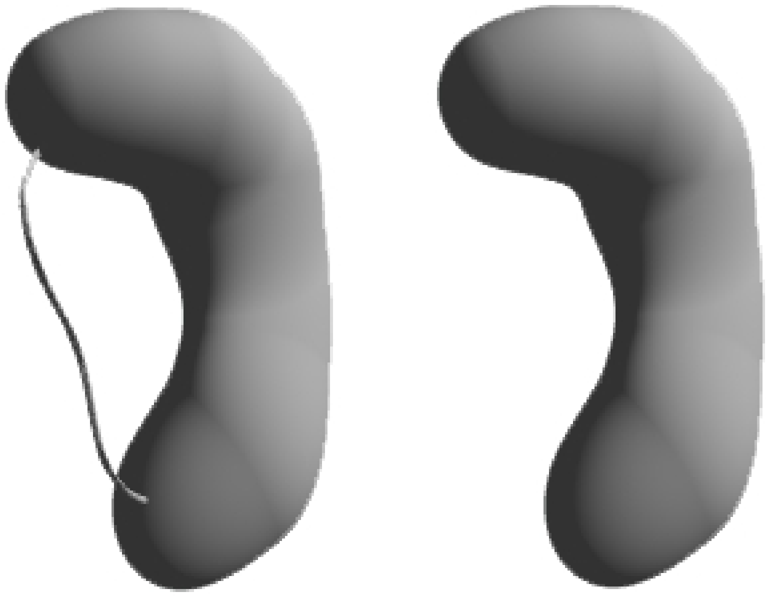

Figure 3.2: Physically indistinguishable membranes with different topologies.

3.1.2 Membrane regularization

We will now establish the relation between the APD algebra of the supermembrane and the $N \to \infty$ limit of a supersymmetric $SU(N)$ matrix model.

We begin by expanding our superspace coordinates into a complete orthonormal set of functions $Y_A(\sigma)$ on the membrane,

$$\vec{X}(\sigma) = \vec{X}_0 + \sum_A \vec{X}^A Y_A(\sigma), \qquad (A = 0, 1, 2, \ldots) \tag{3.28}$$

and an analogous basis for the fermionic coordinates θ. For the sake of simplicity we choose Y_A to be real. We then introduce the metric η_{AB} to enable raising and lowering of $A, B, \ldots$ indices,

$$\eta^{AB} Y_B(\sigma) = Y^A(\sigma), \tag{3.29}$$

where the metric satisfy, as usual,

$$\eta^{AB} \eta_{BC} = \delta^A_C. \tag{3.30}$$

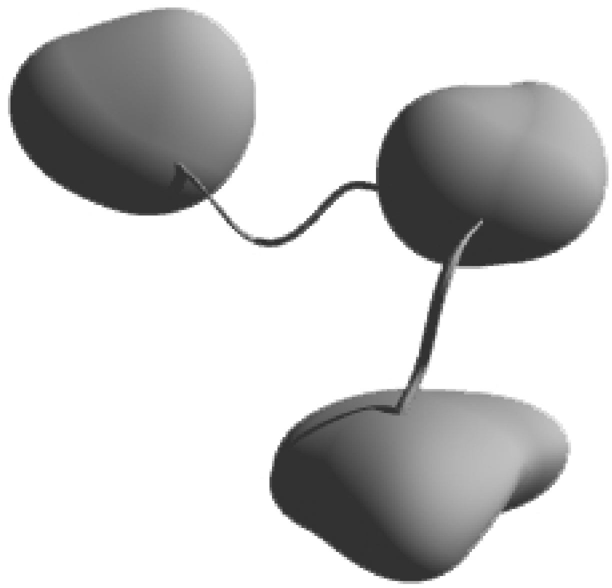

Figure 3.3: Membranes connected by infinitesimally thin tubes.

Normalization of $Y_A(\sigma)$ is done according to the orthogonality relations

$$\int d^2\sigma \sqrt{w(\sigma)} Y_A(\sigma) Y^B(\sigma) = \delta_A^B, \tag{3.31}$$

or, equivalently,

$$\int d^2\sigma \sqrt{w(\sigma)} Y_A(\sigma) Y_B(\sigma) = \eta_{AB}. \tag{3.32}$$

Furthermore we need the completeness relation to be fulfilled,

$$\sum_A Y^A(\sigma) Y_A(\sigma') = \frac{1}{\sqrt{w(\sigma)}} \delta(\sigma - \sigma'). \tag{3.33}$$

This relation is crucial because it allows us to rewrite the Lie bracket in the new basis,

$$\{Y_A, Y_B\} = f_{AB}{}^C Y_C, \tag{3.34}$$

with the totally antisymmetric structure constants

$$f_{AB}{}^C = \int d^2\sigma \, \epsilon^{rs} \partial_r Y_A \partial_s Y_B Y^C. \tag{3.35}$$

To regularize the membrane we now truncate the theory by placing an upper limit Λ on the number of modes indexed by $A, B, \ldots$. The APD group is approximated by a finite-dimensional Lie group G_Λ whose structure constants are equivalent to the APD structure constants in the limit $\Lambda \to \infty$. We then get the consistency condition

$$\lim_{\Lambda\to\infty} f_{AB} \quad C(G_\Lambda) = f_{AB} \quad C(APD), \tag{3.36}$$

for any fixed A, B, C. In the case of spherical membranes [30,32] (a more recent review can be found in [33]) it was shown that

$$G_\Lambda = SU(N) \tag{3.37}$$

where $\Lambda = N^2 - 1$. This result was subsequently generalized to toroidal [34], and then later arbitrary [35], membranes. As we are dealing with $SU(N)$ matrices we can furthermore replace the Lie bracket with a commutator

$$\{\cdot,\cdot\} \to [\cdot,\cdot]. \tag{3.38}$$

We can elucidate the regularization by working through the example of toroidal membranes. We then use the torus coordinates $0 \leq, \sigma_1, \sigma_2 < 2\pi$ and define the basis functions

$$Y_{\vec{m}}(\vec{\sigma}) = e^{i\vec{m}\cdot\vec{\sigma}}, \tag{3.39}$$

where $\vec{m} = (m_1, m_2)$ with m_1 and m_2 being integers. The weight function and metric are, respectively,

$$\sqrt{w(\vec{\sigma})} = \frac{1}{4\pi^2}, \tag{3.40}$$

$$\eta_{mn} = \delta_{m+n}. \tag{3.41}$$

Inserting this metric into the Lie bracket

$$\{A, B\}(\sigma) \equiv \frac{\epsilon^{rs}}{\sqrt{w(\sigma)}}\partial_r A(\sigma)\partial_s B(\sigma), \tag{3.42}$$

will then yield

$$\{Y_{\vec{m}}, Y_{\vec{n}}\} = -4\pi^2(\vec{m}\times\vec{n})Y_{\vec{m}+\vec{n}}. \tag{3.43}$$

This together with the above metric then gives us the structure constants

$$f_{mnk} = -4\pi^2(\vec{m}\times\vec{n})\delta_{m+n+k}. \tag{3.44}$$

Next we use the 't-Hooft clock and shift matrices

$$U = \begin{pmatrix} & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ 1 & & & & \end{pmatrix}, \qquad W = \begin{pmatrix} 1 & & & & \\ & q & & & \\ & & q^2 & & \\ & & & \ddots & \\ & & & & q^{N-1} \end{pmatrix}, \quad (3.45)$$

where

$$q = e^{\frac{2\pi i k}{N}} \quad (3.46)$$

and the matrices satisfy

$$UW = qWU. \quad (3.47)$$

These will now enable us to write any traceless $N \times N$ matrix (and thus all possible $SU(N)$ matrices) as a linear combination of matrices $U^{m_1}W^{m_2}$. The commutator becomes

$$[U^{m_1}W^{m_2}, U^{n_1}W^{n_2}] = (q^{-m_2 n_1} - q^{-m_1 m_2})U^{m_1+n_1}W^{m_2+n_2}. \quad (3.48)$$

We now hold $\vec{m}$ and $\vec{n}$ fixed while we take N to infinity. By Taylor expanding q ($e^x = 1 + x + \mathcal{O}(x^2)$, $x \to 0$ when $N \to \infty$) we get

$$\lim_{N\to\infty} [U^{m_1}W^{m_2}, U^{n_1}W^{n_2}] = \frac{2\pi i k}{N}(\vec{m} \times \vec{n})U^{m_1+n_1}W^{m_2+n_2}. \quad (3.49)$$

From this result we conclude that the $N \to \infty$ limit of $su(N)$ yields the same Lie algebra as area preserving diffeomorphisms on the torus.

An important remark we need to make regards the viable choices of bases of the $SU(N)$ matrices. For the statements we have done on the equivalence between matrix theory and the supermembrane in this section to hold we must choose a particular basis for each membrane topology.

3.1.3 Dimensional reduction of super Yang-Mills theory

We will now make the connection between the supermembrane Hamiltonian and a supersymmetric $SU(N)$ matrix model. One can define the quantum supermembrane as the limit where the truncation of the supersymmetric matrix model is removed. An alternative approach, which we will now discuss, is by dimensional reduction of the maximally supersymmetric $SU(N)$ Yang-Mills theory from $9+1$ to $1+0$ dimensions (for a more thorough review see, e.g., [36])

We start from the 10-dimensional $U(N)$ super Yang-Mills action

$$S = \int d^{10}\xi \left(-\frac{1}{4}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\mathrm{Tr}\bar{\Psi}\Gamma^\mu D_\mu \Psi \right). \quad (3.50)$$

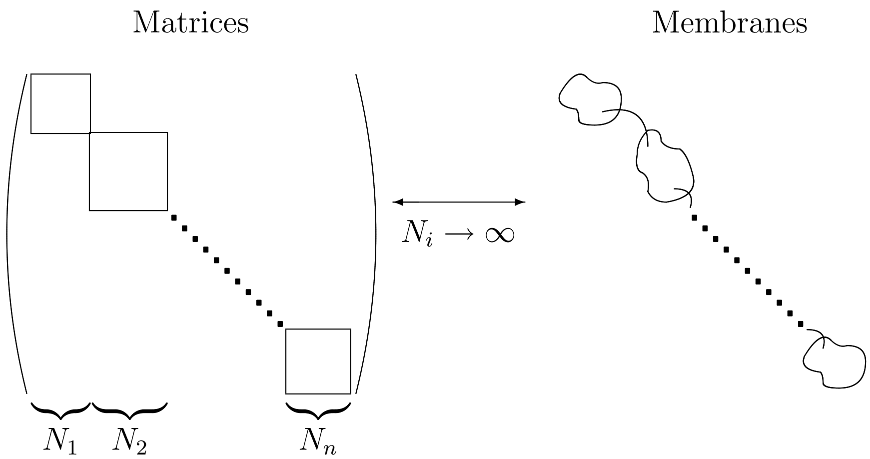

Figure 3.4: Membrane - matrix connection.

The field A_μ is a $U(N)$ hermitian gauge field and Ψ a 16-component Majorana-Weyl spinor of $SO(9,1)$. Both fields are in the adjoint representation of $U(N)$ with adjoint indices suppressed. The field strength is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig_{YM}[A_\mu, A_\nu] \tag{3.51}$$

and measures the curvature of A_μ, with g_{YM} being the Yang-Mills coupling constant. The covariant derivative of Ψ is

$$D_\mu\Psi = \partial_\mu\Psi - ig_{YM}[A_\mu, \Psi]. \tag{3.52}$$

To simplify the forthcoming treatment of the theory we rescale the fields according to

$$A_\mu \rightarrow \frac{1}{g_{YM}} A_\mu \tag{3.53}$$

$$\Psi \rightarrow \frac{1}{g_{YM}} \Psi \tag{3.54}$$

This will cause the coupling constant to appear in the action solely as a multiplicative constant,

$$S = \frac{1}{4g_{YM}^2} \int d^{10}\xi \left(-\mathrm{Tr} F_{\mu\nu}F^{\mu\nu} + 2i\mathrm{Tr}\bar{\Psi}\Gamma^\mu D_\mu \Psi \right), \tag{3.55}$$

with the field strength and the covariant derivative now being

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \tag{3.56}$$

$$D_\mu\Psi = \partial_\mu\Psi - i[A_\mu, \Psi]. \tag{3.57}$$

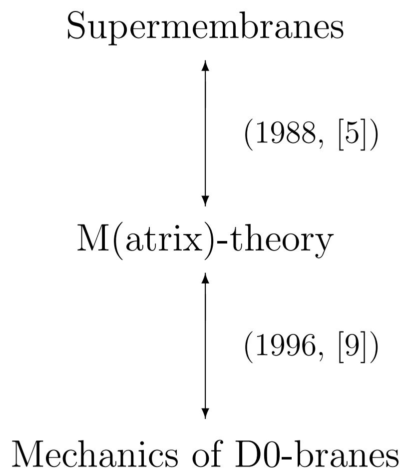

Figure 3.5: Two different approaches to M(atrix) theory.

To proceed with the dimensional reduction we let the 10-dimensional field A_μ decompose into a $(p+1)$-dimensional gauge field A_α and $9-p$ other adjoint scalar fields X^a. With this decomposition we easily derive the dimensionally reduced action

$$S = \frac{1}{4g_{YM}^2} \int d^{p+1}\xi \mathrm{Tr} \left(-F_{\alpha\beta}F^{\alpha\beta} - 2(D_\alpha X^a)^2 + [X^a, X^b] + fermions \right) . \tag{3.58}$$

Before we make the transition to the $1+0$ dimensional theory a few remarks concerning the above action is in order. Besides describing a super Yang-Mills theory in $p+1$ dimensions the above action describes the low energy dynamics of N Dirichlet p-branes (i.e., D-branes) in static gauge (provided that the coupling constant is replaced, of course). D-branes were discovered by Polchinski in 1995. Briefly put, they can be described as topological defects on which open strings can have their endpoints on (for a review, see [37]). From a D-brane viewpoint, A_μ is a gauge field on the D-brane worldvolume and X^a the transverse fluctuations of the D-brane.

We now resume our treatment of the super Yang-Mills action by letting A_μ decompose into nine scalars X^a and a one-dimensional gauge field A_0. By gauging away A_0 we arrive at the Lagrangian,

$$\mathcal{L} = \frac{1}{2} \mathrm{Tr} \left\{ \dot{X}^a \dot{X}^a + \frac{1}{2}[X^a, X^b]^2 + \theta^T (i\dot{\theta} - \Gamma_a [X^a, \theta]) \right\} , \tag{3.59}$$

which then describe a system of N $D0$-branes. From the Lagrangian we then easily derives the corresponding matrix Hamiltonian,

$$H = \frac{1}{2}\mathrm{Tr}\left\{P^a P^a - \frac{1}{2}[X^a, X^b]^2 + \theta^T \Gamma_a [X^a, \theta]\right\}. \tag{3.60}$$

This Hamiltonian is the dimensional reduction of the maximally supersymmetric $SU(N)$ Yang-Mills Hamiltonian from 9+1 to 0+1 dimensions and also the truncated model of the supermembrane. The above Hamiltonian and Lagrangian also play an important role in M-theory by way of the BFSS conjecture, which we will discuss further in section 3.3.5.

3.2 The (super)membrane spectrum

The matter as to whether the bosonic and supersymmetric membrane have continuous or discrete spectra is not without its surprises nor implications, some of which we will discuss now.

3.2.1 The bosonic membrane spectrum

The bosonic Hamiltonian belongs to the group of Hamiltonians where the volume

$$\{(p,q) \mid p^2 + V(q) \leq E\} \tag{3.61}$$

is infinite for some $E < \infty$. For such cases the standard wisdom [2] proclaims that the spectrum is not purely discrete. In the opposite case where the volume is always finite the same wisdom dictates that the spectrum is purely discrete. Wisdom, however, is no match for proper physics and while the latter wisdom holds true, the former does not.

If we express the quantum and classical partition functions as

$$\begin{aligned} Z_q(t) &= \mathrm{Tr}(e^{-tH}) && (3.62)\\ Z_{cl}(t) &= \frac{1}{(2\pi)^\nu}\int d^\nu p\, d^\nu q\, e^{-t(p^2+V(q))}, && (3.63)\end{aligned}$$

we have the Golden-Thompson inequality

$$Z_q(t) \leq Z_{cl}(t). \tag{3.64}$$

Our lightcone membrane Hamiltonian can be re-written [38] and expressed as

$$H = \int d^2\sigma \left(P_i P^i + \sum_{i<j}(X_i X_j)^2\right), \qquad i,j = 1,2,\ldots,D-2. \tag{3.65}$$

If we restrict ourselves to the case of $D = 4$ we obtain (after a slight change in notation) the Hamiltonian

$$H_1 = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + x^2 y^2. \tag{3.66}$$

In [2] no less than five proofs of H_1 having a discrete spectrum are given. We will, however, only concern ourselves with the simplest one. This proof is derived from the zero point harmonic oscillator,

$$-\frac{d^2}{dq^2} + \omega^2 q^2 \geq |\omega| . \tag{3.67}$$

By treating y as a complex number we get

$$-\frac{d^2}{dx^2} + x^2 y^2 \geq |y| . \tag{3.68}$$

By using this and the symmetry between x and y we easily derive the inequality

$$\begin{aligned} H_1 &= -\frac{d^2}{dx^2} - \frac{d^2}{dy^2} + x^2 y^2 & (3.69) \\ &= \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2 y^2\right) + \frac{1}{2}\left(-\frac{d^2}{dy^2} + x^2 y^2\right) \\ &\quad -\frac{1}{2}\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right) & (3.70) \\ &\geq \frac{1}{2}(-\Delta + |x| + |y|) = H_2, & (3.71) \end{aligned}$$

and show that H_2 has a discrete spectrum, since

$$\begin{aligned} \mathrm{Tr}(e^{-tH_2}) &= [1 + \mathcal{O}(1)]\frac{1}{(2\pi)^2}\int d^2p\, dx\, dy\, e^{tp^2 - t|x| - t|y|} & (3.72) \\ &= ct^{-3}[1 + \mathcal{O}(1)]. & (3.73) \end{aligned}$$

This, of course, means that

$$Z_q = \mathrm{Tr}(e^{-tH_1}) \leq ct^{-3}. \tag{3.74}$$

It should be noted, however, that this is a very poor approximation and the true relation [2] should be

$$\mathrm{Tr}(e^{-tH_1}) \leq \mathcal{O}(t^{-3/2}\ln t). \tag{3.75}$$

Nonetheless, our purpose was only to prove the discreteness of the spectrum, which we have now done.

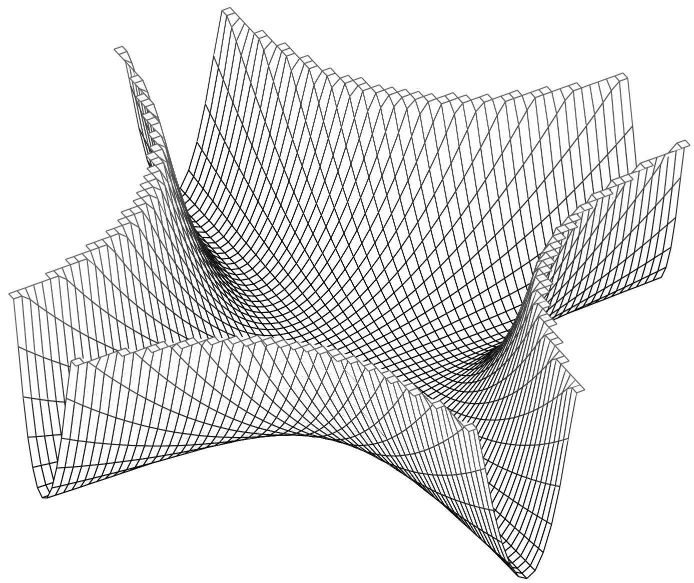

Figure 3.6: x^2y^2 potential valley.

To summarize: classically, wave functions can escape to infinity along the potential valleys. From a membrane perspective this essentially means sprouting string-like objects from the membrane body. This "membrane instability" is cured by quantum mechanics, as the finite-energy wave packets eventually gets stuck in the valleys as these constantly decrease in width.

In regard to the bosonic spectrum some similar work done by Lüscher [39] should be noted. In this paper a discrete spectrum is found for the (non-supersymmetric) $SU(N)$ Yang-Mills theory in 3+1 dimensions. The energy values are expanded in a power series. Also, no ground state was found.

3.2.2 The supermembrane spectrum

In contrast to the bosonic case the supermembrane spectrum is continuous. The proof of this [7] is lengthy and quite technical. Hence we will forsake the full proof in favor of an analogous proof dealing with a simpler toy model. We will use a supersymmetric extension of the Hamiltonian used in the previous section. The Hamiltonian is given by

$$H = \frac{1}{2}\{Q, Q^{\dagger}\}, \tag{3.76}$$

with the supercharges being

$$Q = Q^\dagger = \begin{pmatrix} -xy & i\partial_x + \partial_y \\ i\partial_x - \partial_y & xy \end{pmatrix}, \tag{3.77}$$

with x and y, of course, being the normal Cartesian coordinates. The Hamiltonian then becomes

$$H = \begin{pmatrix} -\Delta + x^2y^2 & x + iy \\ x - iy & -\Delta + x^2y^2 \end{pmatrix}, \tag{3.78}$$

and we immediately recognize the bosonic Hamiltonian in the diagonal elements. The effect of the fermionic parts, however, will be crucial: the off-diagonal terms will make a negative energy contribution and thus negating the confining properties evident in the bosonic theory. More to the point, it will be possible to construct wave packets that can escape to infinity along the coordinate axis (i.e., the potential valleys) without a corresponding infinite cost of energy. The easiest way to show this is to explicitly construct said wave packets.

To proceed, we choose to study the $y = 0$ direction and start with the ansatz

$$\psi_t(x, y) = \chi(x - t)\varphi_0(x, y)\xi_F, \tag{3.79}$$

where $\chi(x)$ is a smooth function with compact support such that χ vanishes unless x is of order t, and

$$\xi_F = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{3.80}$$

If we increase the parameter t the wave packet is translated in the x-direction. Furthermore, as $t \to \infty$ the wave packet moves to infinity along the $y = 0$ valley. The spinor ξ_F was chosen to maximize the negative energy contribution of the wave packet, and we have

$$\xi_F^T H \xi_F = H_{bosonic} - x. \tag{3.81}$$

Moreover, the fermionic contribution to the energy expectation value of the state ψ_t turns out to be $-t + \mathcal{O}(1)$ for large t (χ dominates when t becomes large). This negative contribution is exactly what we need to cancel the bosonic groundstate energy of a harmonic oscillator in the $y = 0$ valley. Next we choose a wave function of such an oscillator,

$$\varphi_0(x, y) = \left(\frac{|\, x \,|}{\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}|x|y^2}. \tag{3.82}$$

For $\nu = 0, 1, 2$ we then have

$$\lim_{t\to\infty}(\psi_t, H^\nu \psi_t) = \int dx\, \chi(x)^*(-\partial_x^2)^\nu \chi(x), \tag{3.83}$$

which is finite. In other words, we are allowed to shift the wave packet to infinity without the energy ever going off to infinity.

To finalize this treatment let us choose an arbitrary energy $E \geq 0$ and $\varepsilon > 0$. Next we choose $\chi(x)$ such that

$$\parallel \chi \parallel = 1, \qquad \parallel (-\partial_x^2 - E)\chi \parallel^2 < \frac{\varepsilon}{2}. \tag{3.84}$$

For large t we will then have

$$\parallel \chi_t \parallel = 1, \qquad \parallel (H - E)\psi_t \parallel^2 < \varepsilon. \tag{3.85}$$

Hence, as ε can be arbitrarily small, we have proved that indeed any value $E \geq 0$ is an energy eigenvalue of the Hamiltonian (3.78), which then have a continuous spectrum. We should end with a remark concerning the full membrane case: here the wave functions can escape to infinity along directions corresponding to the generators of the Cartan subalgebra of the algebra corresponding to the $SU(N)$ group.

3.2.3 A second quantized theory

To recapitulate, we have shown that the bosonic membrane has, classically, a continuous spectrum and, quantum mechanically, a discrete spectrum. If supersymmetry is then switched on the spectrum is again continuous. This is (was) bad news for the first quantization of the theory, which by its very nature should be discrete. In fact, this caused the membrane community to dishearten and disperse into, at the time, more interesting research areas. A few years later, around the time of the BFSS conjecture (see section 3.3.5), there was a revival of membrane theory and the previously fatal flaw, the continuous spectrum, was realized to be no nothing but a blessing in disguise. The simple, yet profound, realization was to interpret the continuous spectrum of the quantum supermembrane not as a hindrance to first quantize the theory, but as a sign that *the theory is second quantized to begin with.* Membrane theory thus deals with "multi-membrane" states from the very outset.

3.3 M-theory

In this section we will try to swiftly cover a large part of what is now called M-theory. The treatment will differ from the rest of the thesis in that it will be of much lesser technical nature. The width of the content is rivaled only by the lack of depth in the treatment.

For more in-depth treatments of string theory, see [40] and [41]; duality, [42]; M-theory, [43] and [42].

3.3.1 Supergravity

Local supersymmetry actually predicts (super)gravity by demanding the existence of the spin 2 graviton and its supersymmetric partner, the spin 3/2 gravitino. In other words, if general relativity had not been discovered at the time of local supersymmetry, we would have been forced to invent it. What's more, supergravity includes the symmetries of both gravity and the grand unified forces, thus making it a candidate for a Theory of Everything. Like we showed for p-branes in chapter 2 supergravity imposes an upper limit of 11 on the number of spacetime dimensions. It is furthermore in this dimensionality that supergravity takes its most elegant form. A serious problem of supergravity, however, is its non-chirality. Nature is chiral, and as Witten among others showed it is impossible to generate a chiral theory from a non-chiral one (ironically, it was Witten who later evaded this no-go theorem). Another problem is the fact that general relativity is non-renormalizable. This in itself is not a disaster, a renormalizable theory containing both massless and massive particles can be disguised as a non-renormalizable theory if we remove the massive particles by using their equations of motion. The remaining non-renormalizable theory containing only massless particles is then fully applicable at energies lower than $m_p c^2$, where m_p is the mass scale associated with the excluded massive particles. If we would want to describe gravity at higher energies, a more fundamental theory with massive particles included would be needed. The mass m_p, we associate with any quantum theory of gravity and is derived from the fundamental constants of gravity (G), special relativity (c) and quantum mechanics (h), thus yielding the relevant energy scale,

$$E_p = m_p c^2 = \sqrt{\frac{hc}{G}} c^2 \approx 10^{16} \text{ TeV}. \tag{3.86}$$

This is the planck energy, and it is in a word, huge. Current energies available at CERN have just reached the TeV range.

In conclusion, the theory we are looking for should contain supersymmetry, massive particles and reduce to Einstein's theory of gravity at low energies. However, we know all the supersymmetric quantum field theories and no one of them fulfill those requirements.

3.3.2 Strings

String first surfaced in theoretical physics in the 1960s as a model of hadrons. The theory suffered from various severe problems and was abandoned in the early 70s in favor of the hugely successful quantum chromodynamics. A smaller group of physicists remained with string theory, however, and eventually managed to solve, sidestep or surmount many of the problems guilty of having condemned string theory to the periphery of respectable research. In addition, string theory was now considered not simply a model of strong interactions, but as a candidate for the Theory of Everything. In 1984, in what has become known as the (first) superstring revolution, string theory entered mainstream theoretical physics. At this time it was shown that certain string theories (there were five) were free of anomalies. In addition, ways to compactify the "excess" dimensions of 10-dimensional superstring theories by way of Calabi-Yau manifolds were also found.

In string theory the length scale is determined by the string tension $T = (2\pi\alpha')^{-1}$, where $\sqrt{\alpha'}$ has dimension of length. For string theory to describe the strength of the gravitational force correctly, we must set

$$\sqrt{\alpha'} \sim 10^{-35} m. \tag{3.87}$$

Consequently, this is also the typical length scale of the strings.

When we want to construct a fully consistent string theory, involving both bosonic and fermionic degrees of freedom, we arrive at no less than five different consistent theories, all in ten spacetime dimensions. They are called the type IIA, type IIB, Type I, $E_8 \times E_8$ heterotic and $SO(32)$ heterotic string theories. We hereby give a short description of each of these theories.

- Type IIA and IIB string theories:. The field description of these two theories contain eight scalar fields (bosons) and sixteen Majorana-Weyl spinors (fermions). Bosonic and fermionic degrees of freedom remains matched as the sixteen Majorana-Weyl spinors are equivalent to eight Majorana spinors. From the chiral nature of the fermions we will differentiate between their handedness by referring to them as left- and right-moving (eight of each). These string theories contain closed strings exclusively, and are thus subject to periodic boundary conditions for the bosonic degrees of freedom. The fermions may have either periodic or anti-periodic conditions, which is referred to as Ramond boundary conditions (R) and Neveu-Schwarz boundary conditions (NS), respectively. Consistency then requires four seperate classes of states in the spectrum: R-R, where both left- and right-moving fermions are subject to periodic boundary conditions. The other sectors are, of course, NS-NS, R-NS and NS-R. The next step is to make a so-called GSO projection. In essence, removing all but about a fourth of the states, keeping the states with an even number of left-moving fermions and an even number of right-moving fermions. The A and B variant of type II string theory arise from the fact that we can choose either an even or odd fermion number to the ground state. In type IIA string theory the GSO projection in the left-moving direction are not the same as in the right-moving direction. In type IIB the GSO projections are identical for both directions.

The supersymmetry algebra of type IIA is the non-chiral $N = 2$ superalgebra, while type IIB have the chiral $N = 2$ superalgebra. Both consist of 32 supersymmetry generators.

- The heterotic string theories: As with type II string theories these are closed and oriented strings. Although they both have eight scalar fields, they have, unlike the type II string, eight right-moving Majorana-Weyl fermions and 32 left-moving Majorana-Weyl fermions. Heterotic strings are divided into sectors according to their Neveu-Schwarz and Ramond boundary conditions and then GSO projected, in a similar (but not identical) way to type II strings. The resulting consistent string theories are the $SO(32)$ heterotic and $E_8 \times E_8$ heterotic string theories; their names reflecting their respective gauge groups. Their common superalgebra is the $N = 1$ chiral supersymmetry algebra, which has 16 real generators.
- Type I string theory: In contrast to the other four string theories, the type I theory contains unoriented, both open and closed, strings. Open strings of course cannot have periodic boundary conditions and are instead subjected to Neumann boundary conditions. Furthermore is type I string theory invariant under world-sheet parity transformation, a symmetry that exchanges left- and right-moving sectors. Like the heterotic string, the type I spectrum is invariant under the $N = 1$ chiral superalgebra.

3.3.3 Duality

Duality is an extremely valuable tool and was to a large part responsible of unifying the five different string theories under the common banner of M-theory. What we have said previously about string theory was in regard of the perturbative regime only. However, duality manages to bridge the gap to the non-perturbative region. We will touch briefly on the three principal dualities:

- T-duality: a perturbative duality relating distances.
- S-duality: a non-perturbative duality relating strong and weak coupling.
- U-duality: a duality of dualities relating T- and S-duality to each other.

If we compactify a string theory on a circle of radius R we would get the normal so-called Kaluza-Klein modes quantized according to

$$p = \frac{n}{R}, \tag{3.88}$$

where n is an integer. However, as we are dealing with strings and not particles we will have additional modes corresponding to a closed string wrapped around the circle, with modes

$$p = mR, \tag{3.89}$$

where m is the number of times the string has wound around the circle. Ergo, the left- and right-moving modes are

$$(p_L, p_R) = \left(\frac{n}{2R} + mR, \frac{n}{2R} - mR\right). \tag{3.90}$$

Thus the mass spectrum for $\mathcal{M}^2$ is invariant under

$$R \longleftrightarrow \frac{1}{2R}, \tag{3.91}$$

if we also make the exchange $n \leftrightarrow m$. Hence the string seem to exhibit the same physics if dimensionally reduced to a circle of minuscule size as to one of titanic proportions. This invariance is the essence of T-duality and in fact yields an equivalence between type IIA and type IIB string theory as well as an equivalence between the two heterotic strings.

S-duality, on the other hand, relates strong coupling to weak coupling, and vice versa. As such, it is a very powerful tool and enables one to use perturbation theory to probe previously barred regions of a theory. More importantly, S-duality, gives a correspondence between 10-dimensional string theories and 11-dimensional M-theory (the name, M-theory, have been rather abused and can mean either the 11-dimensional non-perturbative extension of string theory, or the entire theory formerly known as "string theory"; this thesis will abuse the name further and use it in both cases interchangeably and letting the context decide its meaning). M-theory is S-dual to type IIA string theory and its strong coupling limit ($g_s \to \infty$) when M-theory is compactified on a circle, S_1 [4] (see also [44]). Furthermore, M-theory reduces to the $E_8 \times E_8$ heterotic string when compactified on a line segment, $S_1/\mathbb{Z}_2$. S-duality also connects the SO(32) heterotic string with the type I string. A special case of S-duality concerns type IIB, which in fact is self-dual and thus relates the different regions of the moduli space within the same theory.

The third important duality is U-duality, which connects T- and S-duality to each other. If a string exhibits T-duality the U-dual string would exhibit S-duality, and vice versa. This put the, at the time, conjectured S-duality on a firm footing as it was shown to be equivalent to the well-established T-duality.

3.3.4 M-theory

What has become known as the second superstring revolution was ignited by a celebrated talk in 1995 by Edward Witten. The five string theories were now firmly unified by the different dualities. A M(other) theory living in 11-dimensional spacetime with 11 dimensional supergravity as the low-energy limit and interconnected with all the five string theories now became the accepted starting point for further research. M-theory (in the broad sense of the word) was the name given to this vast theory, and is summarized in figure 3.7. The link between type I and type IIB, in the figure denoted by Ω, is an "orientifold projection". Going from type IIB we make an orientation reversal, i.e., $\sigma \to -\sigma$. When the smoke clears we are left with unoriented closed strings and unoriented open strings with an $SO(32)$ gauge symmetry: Type I string theory.

3.3.5 The BFSS conjecture

Despite the properties we have mentioned M-theory must have, any concrete formulation of the theory has not been done. An attempt at this was made by Banks, Fishler, Shenker and Susskind (BFSS) in 1996 with a bold conjecture [45] suggesting that, "M-theory, in the lightcone frame, is exactly described by the large N limit of a particular supersymmetric matrix quantum mechanics". More specifically, the matrix quantum mechanics consists of N $D0$-branes. From section 3.1.3 we know that the dynamics of such a system of branes are described by the dimensionally reduced $U(N)$ super Yang-Mills Hamiltonian from $9+1$ to $0+1$ dimensions. A D-brane has the mass

$$M = \frac{m_s}{g_s}, \tag{3.92}$$

where m_s is the mass scale of type IIA string theory. Having compactified the 11-dimensional theory on a circle of radius R, the following relation

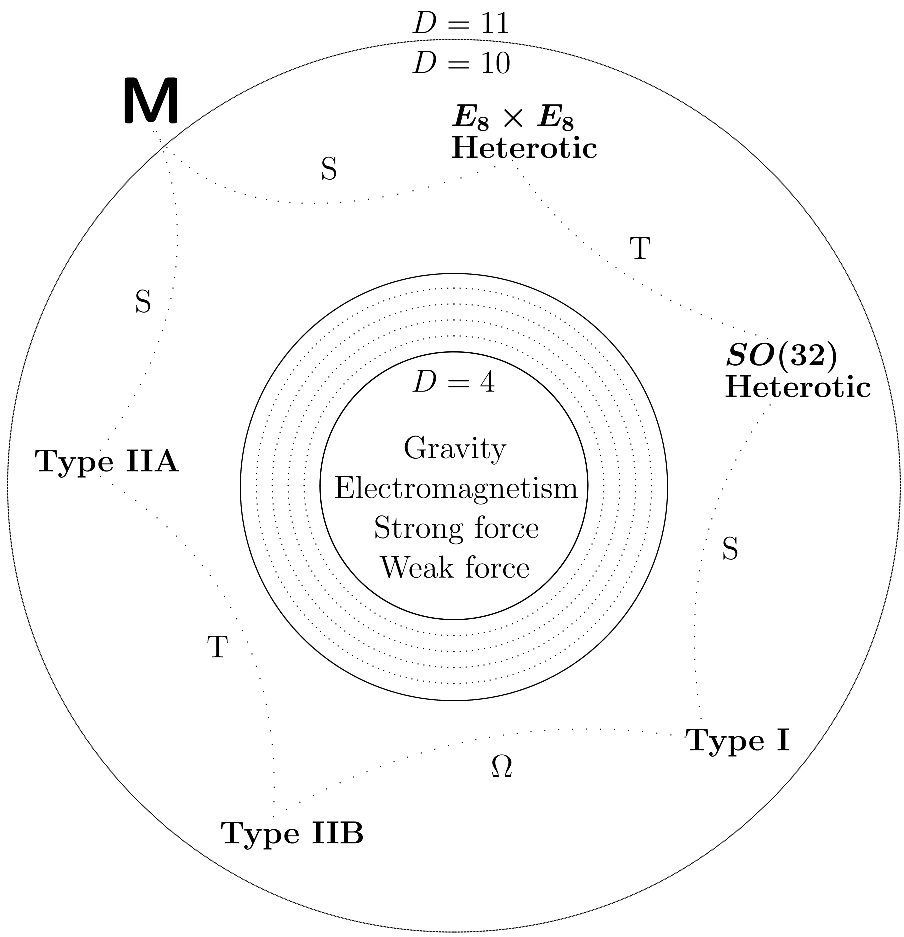

Figure 3.7: M-theory schematics.

holds (up to numerical pre-factors),

$$g_s = Rm_s. \tag{3.93}$$

We thus deduce that $M = 1/R$, which prompts us to interpret the brane as the first Kaluza-Klein excitation of the 11-dimensional supergravity multiplet on the circular dimension. The spatial coordinates of the system of $D0$-branes are then represented by $N \times N$ matrices. There might be higher order corrections, but these are then suppressed by going to the infinite momentum frame. This frame is reached by letting the momentum

$$p_{11} = \frac{N}{R} \tag{3.94}$$

go to infinity at the same time as R does so. The BFSS conjecture is that M-theory is described exactly and non-perturbatively by the $N \to \infty$ infinite momentum frame of N $D0$-branes. The $N \to \infty$ limit is of course troublesome and perhaps also unnecessary; Susskind made a further conjecture [46] in 1997, suggesting that the BFSS conjecture was valid even for finite N.

Since its birth the BFSS conjecture has received much attention and been intensely scrutinized. More evidence for its validity has been presented, but perhaps more importantly, so has evidence for the opposite. For instance, in the aptly named article "Why matrix theory is hard" [47] the authors show a possible contradiction arising from the BFSS conjecture. At any rate, the last word has not been said and matrix theory remains an exciting topic in M-theory.

For further information and references on matrix theory and the BFSS conjecture, see [48] and [49].

4

The Membrane Vacuum State

In this chapter we will mainly investigate the ground state of a supersymmetric $SU(2)$ matrix model. This is a natural starting point in the search for the membrane vacuum state due to the connection between the supermembrane and the large N limit of $SU(N)$ matrix models. The bulk of the treatment will deal with such an $SU(2)$ invariant ground state, whose asymptotic form we will derive; asymptotic in this case meaning far away from the center of the potential.

4.1 Overview and preliminaries

This section will present a short chronological overview of the research that has been done regarding membrane ground states thus far. As a precursor to the higher-dimensional case presented later we will also apply our method of finding a vacuum state on a two-dimensional toy model.

4.1.1 Current state of affairs

The search for the membrane vacuum state spans about twenty years of research and a large number of articles. We will here attempt to outline the research done (and not done) so far. Invariably, this will for the most part be a review of the *tour de force* of Jens Hoppe and collaborators: [5, 50, 51, 52, 53, 54, 55, 10, 56, 57, 58, 59, 60, 61].

In 1984, Claudson and Halpern [6] published their work on supersymmetric ground state wave functions where they investigated the viability of constructing such states for many different kinds of supersymmetric Hamiltonians. The first attempt at constructing a ground state wave function specifically for the membrane was conducted by de Wit, Hoppe and Nicolai in their seminal paper [5] in 1988. Their analysis concerned using truncations of the supermembrane. They showed for two different truncations the absence of a zero energy ground state. They concluded that, while leaning towards a massive ground state, more work on the subject was needed. Quite a few years later [50] Hoppe slightly extended this earlier work, by commentating on some subtleties in the original paper and also showing the lack of a (real) zero energy ground state in the simplest case of a supermembrane matrix model ($SU(2)$ gauge invariant supermembrane in $D = 4$ spacetime). Then in a subsequent paper [51] it was firmly proven that a normalizable zero mass ground state did not exist in the $SU(2)$, $D = 4$ case. Later the same year Hoppe, in a trio of papers published at a machine-gun pace, made use of another method to approach the ground state problem. In the first paper [52] he proves that no solution exists for the case where $D = 4$, $SU(N)$ with N odd. He also outlines an analogous model for the more interesting case of 11 dimensions, which is continued in the following paper by obtaining *bosonic* solutions to the equations $Q^{\dagger}_{\beta}\psi = 0$. The third paper further investigates the 11-dimensional case and tentatively constructs zero energy states. A few months later, in the end of 1997, Halpern and Schwartz [62] used a generalized Hamiltonian Born-Oppenheimer formulation to derive asymptotic, normalizable zero energy solutions of the Schrödinger equation of $SU(2)$ matrix theory corresponding to the 11-dimensional supermembrane. Among the ground state candidates found one was $Spin(9)$ invariant, a property whose significance will become clear later. The result obtained by their rather cumbersome calculation was later reproduced by Graf and Hoppe [55] with a simpler approach using the supercharges (containing first order derivatives) instead of the Hamiltonian (containing second order derivatives). The asymptotic ground state was $SU(2)$ and $SO(9)$ invariant and determined to leading and sub-leading order from a perturbative expansion of the 16 supercharges $Q_\beta = Q^{(0)}_\beta + Q^{(1)}_\beta + \ldots$. In a follow-up paper [10] in 1999 a more general theorem was given regarding the states satisfying $Q_\beta\psi = 0$ of the supersymmetric matrix models corresponding to supermembranes in dimensions $D = 4, 5, 7, 11$. It is explicitly shown that the 11-dimensional case is the only case with a square-integrable $SU(2) \times Spin(D-2)$ invariant asymptotic ground state wave function, the form of which was also given. The theorem along with its proof will be treated in detail in section 4.2 and 4.3, respectively. Some months after this paper Hoppe and collaborators [56] started to investigate the $SU(3)$ invariant zero energy state, an investigation continued in [57] and culminated in [58].

In 2001, Graf, Hasler and Hoppe [59] resolved the issue whether the supersymmetric x^2y^2 potential allows for zero energy ground states or not. A similar calculation will also be presented in the next section. In 2002, striving to go beyond the $SU(2)$ and $SU(3)$ case, Hasler and Hoppe attempt in [60] to generalize the previous model to include $SU(N)$ invariance. They show that, for all $N > 1$, when the eigenvalues of the matrices in the matrix model become large and well separated from each other the vacuum state wave function factorizes into a product of supersymmetric harmonic oscillator wave functions and an additional wave function that will be annihilated by a certain supercharge. Then in the last relevant paper [61] to date, the same authors prove that the zero energy states for reduced super Yang-Mills theory in $d+1$ dimensions with $d = 2, 3, 5, 9$ (the dimension d corresponds to a supermembrane living in D dimensions according to $D = d+2$) is necessarily $Spin(d)$ invariant.

To summarize the current state of this field, we quote this last paper: "The general belief, partially proven, is that for $d = 2, 3, 5$ no zero energy state exists and that for $d = 9$ there exists a unique ground state".

4.1.2 A toy model ground state

In this section we will investigate the possibility of normalized ground state of a simple two-dimensional model. The supercharge and Hamiltonian are similar to the ones used in the toy model of the supermembrane; they are,

$$Q = i \begin{pmatrix} \partial_x & \partial_y + xy \\ \partial_y - xy & -\partial_x \end{pmatrix}, \tag{4.1}$$

and

$$H = (-\partial_x^2 - \partial_y^2 + x^2y^2)\mathbb{1} + \begin{pmatrix} x & -y \\ -y & -x \end{pmatrix}, \tag{4.2}$$

respectively. We are by now familiar with the characteristics and implications of the bosonic potential, x^2y^2, and hence abstain from any further comments on the potential. It should be duly noted, however, that while the Hamiltonian is simple, the question whether it admits a normalizable ground state, is not. In fact, this question remained unsolved for more than ten years and was given a resolution only recently. The argument is the following; an approximate solution to

$$Q\Psi = 0, \tag{4.3}$$

when $x \to +\infty$ is

$$\Psi_0 = e^{-\frac{xy^2}{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{4.4}$$

We use this function as the first term in the asymptotic expansion

$$\Psi = x^{-\kappa}(\Psi_0 + \Psi_1 + \ldots). \tag{4.5}$$

We also expand the supercharge in powers of x, as

$$Q = \sum_{n=0}^{\infty} Q^{(n)}, \tag{4.6}$$

where n denotes the different powers of x. In our toy model case the supercharge (4.1) is just the sum of two terms. We have,

$$Q^{(0)} = i\begin{pmatrix} \partial_x & 0 \\ 0 & -\partial_x \end{pmatrix} \tag{4.7}$$

$$Q^{(1)} = i\begin{pmatrix} 0 & \partial_y - xy \\ \partial_y - xy & 0 \end{pmatrix}. \tag{4.8}$$

Equation (4.3) now becomes

$$Q\Psi = \left(\begin{pmatrix} 0 & \partial_y + xy \\ \partial_y - xy & 0 \end{pmatrix} + \begin{pmatrix} \partial_x & 0 \\ 0 & -\partial_x \end{pmatrix}\right)\left(x^{-\kappa}(\Psi_0 + \ldots)\right) = 0$$

This equation is equivalent to a set of equations for $n = 0, 1, \ldots$. By matching powers of x we get,

$$\begin{pmatrix} 0 & \partial_y + xy \\ \partial_y - xy & 0 \end{pmatrix}\Psi_n + x^{\kappa}\begin{pmatrix} \partial_x & 0 \\ 0 & -\partial_x \end{pmatrix} x^{-\kappa}\Psi_{n-1} = 0, \tag{4.9}$$

with $n = 1, 2, \ldots$, and where we multiplied the equation by x^{κ}. We immediately notice that the case where $n = 0$ is missing, yielding the additional equation

$$\begin{pmatrix} 0 & \partial_y + xy \\ \partial_y - xy & 0 \end{pmatrix}\Psi_0 = 0, \tag{4.10}$$

where we again have multiplied the equation with x^{κ}. It is now a simple matter to check that equation (4.10) holds (it does so trivially). To check equation (4.9) we first multiply the equation with $\Psi_0^{\dagger}$ from the left. This kills the first term as we let the derivatives it contain act to the left on $\Psi_0^{\dagger}$. After integrating over y the remaining term becomes,

$$\int_{-\infty}^{\infty} (0, e^{-\frac{xy^2}{2}}) x^{\kappa}\begin{pmatrix} \partial_x & 0 \\ 0 & -\partial_x \end{pmatrix} x^{-\kappa}\Psi_{n-1}dy. \tag{4.11}$$

This simplifies to

$$-\int_{-\infty}^{\infty} e^{-\frac{xy^2}{2}} x^{\kappa}\partial_x x^{-\kappa}\Psi_{n-1}dy. \tag{4.12}$$

We now check the case of $n = 1$, which yields

$$- \int_{-\infty}^{\infty} \left(\frac{\kappa}{x} + \frac{y^2}{2} \right) e^{-xy^2} dy = 0, \tag{4.13}$$

which have the only solution

$$\kappa = -\frac{1}{4}. \tag{4.14}$$

Moreover, we note that

$$\int_{-\infty}^{\infty} | x^{\frac{1}{4}} e^{-\frac{xy^2}{2}} |^2 \, dx = \infty. \tag{4.15}$$

This result proves conclusively that the Hamiltonian (4.2) does not allow for any *square-integrable* solution of the asymptotic form (4.5).

From (4.9) we may calculate $\Psi_{n>0}$,

$$\Psi(x, y) = x^{\frac{1}{4}} e^{-\frac{xy^2}{2}} \sum_{m=0}^{\infty} x^{-\frac{3m}{2}} \begin{pmatrix} \frac{y}{4x} f_m(xy^2) \\ g_m(xy^2) \end{pmatrix}, \quad x \to \infty, \tag{4.16}$$

where $f(s)$ and $g(s)$ are the unique polynomial solutions of

$$8g'_{m+2} = \left(\frac{3}{4} + \frac{3m}{2} + \frac{s}{2} \right) f_m - s f'_m, \tag{4.17}$$

$$2s f'_m + (1 - 2s) f_m = (1 - 2s - 6m) g_m + 4s g'_m, \tag{4.18}$$

and the starting values,

$$f_0 = g_0 = 1 \quad , \tag{4.19}$$

$$f_1 = g_1 = 0 \quad . \tag{4.20}$$

The validity of (4.17) and (4.18) is best shown by inserting (4.16) into the equation (4.9).

4.2 The $SU(2)$ ground state theorem

This section presents the notation and model used in the $SU(2)$ ground state theorem, which is also formulated.

4.2.1 The model

If we have $SU(2)$ as the gauge group the model we are using can be interpreted as a supersymmetric quantum mechanical system of d particles in $X = \mathbb{R}^3$ space.

Using the notation found in [10] we have the bosonic coordinates

$$q = (\vec{q}_1, \dots, \vec{q}_d) = (q_{sA})_{s=1,\dots,d;\ A=1,2,3.} \tag{4.21}$$

Their fermionic counterparts are

$$\gamma^i = (\gamma^i_{\alpha\beta})_{i=1,\dots,d;\ \alpha,\beta=1,\dots,s_d.} \tag{4.22}$$

Here s_d is the smallest dimension of the real representation of the Clifford algebra with d generators,

$$\{\gamma^s, \gamma^t\} = 2\delta^{st}. \tag{4.23}$$

The relation between s_d and d is

$$s_d = \begin{cases} 2^{[d/2]} & d = 0,1,2 \mod 8 \\ 2^{[d/2]+1} & \text{otherwise,} \end{cases} \tag{4.24}$$

with $[\cdot]$ being the integer part only. For the dimensionalities of interest we thus have

d	s_d
2	2
3	4
5	8
9	16

(4.25)

We can realize $Spin(d)$ in the representation space through matrices R of $SO(s_d)$, in other words,

$$Spin(d) \hookrightarrow SO(s_d). \tag{4.26}$$

We will also need a Clifford algebra with s_d generators and its irreducible representation on $\mathcal{C} = \mathbb{C}^{2^{s_d/2}}$. Incorporating the three "spatial" degrees of freedom we then arrive at $\mathcal{C}^{\otimes 3}$, with the Clifford generators

$$(\vec{\Theta}_1, \dots, \vec{\Theta}_{s_d}) = (\Theta_{\alpha A})_{\alpha=1,\dots,s_d;\ A=1,2,3.} \tag{4.27}$$

satisfying the Clifford algebra

$$\{\Theta_{\alpha A}, \Theta_{\beta B}\} = \delta_{\alpha\beta}\delta_{AB}, \tag{4.28}$$

which, if realized explicitly, should be done so with great care (we will do this for the $d = 3$ case in the appendix). The Hilbert space is thus

$$\mathcal{H} = L^2(X, \mathcal{C}^{\otimes 3}) \tag{4.29}$$

The supercharges that will act on the Hilbert space are

$$Q_\beta = \vec{\Theta}_\alpha \cdot (-i\gamma^t_{\alpha\beta}\vec{\nabla}_t + \frac{1}{2}\vec{q}_s \times \vec{q}_t \gamma^{st}_{\beta\alpha}), \tag{4.30}$$

where $\vec{\nabla}_t$ are simply the partial derivatives w.r.t. $\vec{q}_t$ and with $\alpha, \beta = 1, \ldots, s_d$ and $s, t = 1, \ldots, d$, and where,

$$\gamma^{st} = \frac{1}{2}(\gamma^s\gamma^t - \gamma^t\gamma^s). \tag{4.31}$$

The supercharges transform as scalars under $SU(2)$ gauge transformations that are generated by

$$J_{AB} = L_{AB} + M_{AB} \equiv -i(q_{sA}\partial_{sB} - q_{sB}\partial_{sA}) - \frac{i}{2}(\Theta_{\alpha A}\Theta_{\alpha B} - \Theta_{\alpha B}\Theta_{\alpha A}) \tag{4.32}$$

and as spinors under the $Spin(d)$ transformations generated by

$$J_{st} = L_{st} + M_{st} \equiv -i(\vec{q}_s \cdot \vec{\nabla}_t - \vec{q}_t \cdot \vec{\nabla}_s) - \frac{i}{4}\vec{\Theta}_\alpha \gamma^{st}_{\alpha\beta}\vec{\Theta}_\beta, \tag{4.33}$$

To obtain the Hamiltonian of the system we need to calculate the anticommutation relations of the supercharges. Before we proceed with this task a warning to sensitive souls and mathematicians is in order; the following treatment will exhibit a certain laxness of index positioning. The inherent notation[1] of the various quantities involved occasionally make it highly undesirable to be too meticulous about those lesser things in life. It should be strongly stressed, however, that clarity in the presentation have not been sacrificed in favor of aesthetics (there are no aesthetic expressions in this chapter). If despite this someone would find the following calculation hard to digest, please take a look at the original work [10] before voicing any complaints. It should also be noted that the calculation is done for the case where $d = 9$.

Having thus dispensed with the amenities, we go right back to work. Anticommuting the supercharges immediately result in the following four

[1] We follow the original notation used in [10].

terms,

$$
\begin{aligned}
\{Q_\alpha, Q_\beta\} &= \left\{\vec{\Theta}_\gamma \cdot (-i\gamma^t_{\gamma\alpha}\vec{\nabla}_t), \vec{\Theta}_\delta \cdot (-i\gamma^p_{\delta\beta}\vec{\nabla}_p)\right\} \\
&\quad + \left\{\vec{\Theta}_\gamma \cdot \frac{1}{2}\vec{q}_s \times \vec{q}_t \gamma^{st}_{\alpha\gamma}, \vec{\Theta}_\delta \cdot \frac{1}{2}\vec{q}_p \times \vec{q}_q \gamma^{pq}_{\beta\delta}\right\} \\
&\quad + \left\{\vec{\Theta}_\gamma \cdot (-i\gamma^t_{\gamma\alpha}\vec{\nabla}_t), \vec{\Theta}_\delta \cdot \frac{1}{2}\vec{q}_p \times \vec{q}_q \gamma^{pq}_{\beta\delta}\right\} \\
&\quad + \left\{\vec{\Theta}_\gamma \cdot \frac{1}{2}\vec{q}_s \times \vec{q}_t \gamma^{st}_{\alpha\gamma}, \vec{\Theta}_\delta \cdot (-i\gamma^p_{\delta\beta}\vec{\nabla}_p)\right\} \\
&= I + II + III + IV.
\end{aligned}
$$

Starting with the first term, we have,

$$
\begin{aligned}
I &= -\gamma^t_{\alpha\gamma}\gamma^p_{\beta\delta}\nabla_{tA}\nabla_{pB}\{\Theta^{\gamma A}, \Theta^{\delta B}\} \\
&= -\left(\gamma^{(t}\gamma^{s)}\right)_{\alpha\beta}\nabla_{tA}\nabla_{pB}\,\delta^{AB} \\
&= -\frac{1}{2}\{\gamma^t, \gamma^p\}_{\alpha\beta}\nabla_{tA}\nabla_{pB}\,\delta^{AB} = -\delta_{\alpha\beta}\sum_{s=1}^{9}\vec{\nabla}^2_s,
\end{aligned}
\tag{4.34}
$$

where we have simply used the Clifford algebras of $\Theta^{\alpha A}$ and γ^s , respectively.

At this juncture we should also mention the symmetry/antisymmetry of the γ-matrices, vital properties we will use consistently throughout our calculations,

$$
\begin{array}{ll}
\gamma^{(0)}, & \gamma^{(1)} \text{ symmetric} \\
\gamma^{(2)}, & \gamma^{(3)} \text{ antisymmetric} \\
\gamma^{(4)}, & \gamma^{(5)} \text{ symmetric} \\
\gamma^{(6)}, & \gamma^{(7)} \text{ antisymmetric} \\
\gamma^{(8)}, & \gamma^{(9)} \text{ symmetric}
\end{array}
$$

The second term becomes,

$$
\begin{aligned}
II &= \frac{1}{4}\gamma^{st}_{\alpha\gamma}\gamma^{pq}_{\beta\delta}\epsilon^{ABC}q_{sB}q_{tC}\epsilon^{DEF}q_{pE}q_{qF}\{\Theta_{\gamma A}, \Theta_{\delta D}\} \\
&= -\frac{1}{4}\left(\gamma^{st}\gamma^{pq}\right)_{(\alpha\beta)}\epsilon^{ABC}q_{sB}q_{tC}\epsilon_A{}^{EF}q_{pE}q_{qF} \\
&= -\frac{1}{4}\left(\gamma^{st}\gamma^{pq}\right)_{(\alpha\beta)}\delta^{BC}_{EF}\,q_{sB}q_{tC}q_{pE}q_{qF} \\
&= -\frac{1}{4}\left(\left(\gamma^{stpq}\right)_{(\alpha\beta)} - 4\delta^{[s}_{[p}(\gamma^{t]}{}_{q]})_{(\alpha\beta)} - 2\delta^{st}_{[pq]}\delta_{\alpha\beta}\right)\delta^{BC}_{EF}\,q_{sB}q_{tC}q_{pE}q_{qF}.
\end{aligned}
\tag{4.35}
$$

Of the three terms above, only the last one survives. The first is zero due to the antisymmetry in $BCEF$ and symmetry of δ, the second due to symmetry in $\alpha\beta$ coupled with the antisymmetry of $\gamma^{(2)}$. To continue, we now have

$$\begin{aligned} \ldots &= -\frac{1}{4}\delta_{\alpha\beta}\,\delta^{BC}_{EF}(q_{sB}q_{tC}q_{tE}q_{sF} - q_{sB}q_{tC}q_{sE}q_{tF}) \\ &= \delta_{\alpha\beta}\,\frac{1}{2}(\epsilon^{ABC}q_{sB}q_{tC})^2 = \delta_{\alpha\beta}\sum_{s<t}(\vec{q}_s \times \vec{q}_t)^2. \end{aligned} \tag{4.36}$$

To proceed with III and IV we need to calculate two terms containing derivatives. Term III is

$$\begin{aligned} &-\frac{i}{2}\left(\Theta_\gamma{}^A\Theta_\delta{}^D\gamma^t_{\gamma\alpha}\gamma^{pq}_{\beta\delta}\nabla_{tA}\epsilon_{DEF}q_p{}^Eq_q{}^F + \Theta_\delta{}^D\Theta_\gamma{}^A\gamma^{pq}_{\beta\delta}\gamma^t_{\gamma\alpha}\epsilon_{DEF}q_p{}^Eq_q{}^F\nabla_t{}^A\right) \\ &= -\frac{i}{2}\left(\Theta_\gamma{}^A\Theta_\delta{}^D\gamma^t_{\gamma\alpha}\gamma^{pq}_{\beta\delta}\epsilon_{DEF}\left(\delta_{tp}\,\delta^{AE}q_q{}^F + \delta_{tq}\,\delta^{AF}q_p{}^E + q_p{}^Eq_q{}^F\nabla_t{}^A\right)\right. \\ &\quad \left. + \Theta_\delta{}^D\Theta_\gamma{}^A\gamma^{pq}_{\beta\delta}\gamma^t_{\gamma\alpha}\epsilon_{DEF}q_p{}^Eq_q{}^F\nabla_t{}^A\right). \end{aligned} \tag{4.37}$$

In the next step we will use the simple relation

$$AB = \frac{1}{2}[A,B] + \frac{1}{2}\{A,B\}. \tag{4.38}$$

Contracting indices and using the antisymmetry of ϵ_{DEF} we obtain,

$$\begin{aligned} III &= -\frac{i}{2}\left(\Theta^{[E}_{(\gamma}\Theta^{D]}_{\delta)}\gamma^t_{\gamma\alpha}\gamma^{tq}_{\beta\delta}\epsilon_{DEF}q_q{}^F\right. \\ &\quad + \Theta^{[F}_{(\gamma}\Theta^{D]}_{\delta)}\gamma^t_{\gamma\alpha}\gamma^{tq}_{\beta\delta}\epsilon_{DEF}q_p{}^E \\ &\quad + \Theta^{[A}_{(\gamma}\Theta^{D]}_{\delta)}\gamma^t_{\gamma\alpha}\gamma^{pq}_{\beta\delta}\epsilon_{DEF}q_p{}^Eq_q{}^F\nabla_t{}^A \\ &\quad + \frac{1}{2}\gamma^t_{\delta\alpha}\gamma^{pq}_{\beta\delta}\epsilon_{DEF}q_p{}^Eq_q{}^F\nabla_t{}^D \\ &\quad \left. + \;\Theta_\delta{}^D\Theta_\gamma{}^A\gamma^{pq}_{\beta\delta}\gamma^t_{\gamma\alpha}\epsilon_{DEF}q_p{}^Eq_q{}^F\nabla_t{}^A\right), \end{aligned} \tag{4.39}$$

and a very similar result for term IV. Taking the non-Θ-terms of III and IV together and renaming dummy indices, we get

$$-\frac{i}{4}\left(\gamma^p_{\delta\beta}\gamma^{st}_{\alpha\delta} + \gamma^p_{\delta\alpha}\gamma^{st}_{\beta\delta}\right)\epsilon_{ABC}q_s{}^Bq_t{}^C\nabla_p{}^A. \tag{4.40}$$

By manipulating the Dirac matrices according to,

$$\begin{aligned} \gamma^p_{\delta\beta}\gamma^{st}_{\alpha\delta} + \gamma^p_{\delta\alpha}\gamma^{st}_{\beta\delta} &= \left(\gamma^{st}\gamma^p - \gamma^p\gamma^{st}\right)_{\alpha\beta} \\ &= \frac{1}{2}\left(\gamma^s\gamma^t\gamma^p - \gamma^t\gamma^s\gamma^p - \gamma^p\gamma^s\gamma^t + \gamma^p\gamma^t\gamma^s\right)_{\alpha\beta} \\ &= 2\left(\delta^{pt}\gamma^s - \delta^{sp}\gamma^t\right)_{\alpha\beta}, \end{aligned} \tag{4.41}$$

and inserting this result into (4.40) we get,

$$-\frac{i}{2}\gamma^t_{\alpha\beta}q_t{}^A\epsilon_{ABC}\left(q_s{}^B\nabla_s{}^C - q_s{}^C\nabla_s{}^B\right). \tag{4.42}$$

Saving this for later and instead collecting the remaining terms in III and IV containing derivatives we are faced with these two creatures,

$$-\frac{i}{2}\Theta_\gamma{}^A\Theta_\delta{}^D\left(\gamma^{st}_{\alpha\gamma}\gamma^p_{\delta\beta} + \gamma^{st}_{\beta\gamma}\gamma^p_{\delta\alpha}\right)\epsilon_{ABC}q_s{}^B q_t{}^C\nabla_{pD} \tag{4.43}$$

$$-\frac{i}{2}\Theta^{[D}_{(\delta}\Theta^{A]}_{\gamma)}\left(\gamma^{st}_{\alpha\gamma}\gamma^p_{\delta\beta} + \gamma^{st}_{\beta\gamma}\gamma^p_{\delta\alpha}\right)\epsilon_{ABC}q_s{}^B q_t{}^C\nabla_{pD}. \tag{4.44}$$

Again using (4.38) and cancelling some terms, we end up with

$$\begin{aligned} &-\frac{i}{4}\left(\gamma^{st}_{\alpha\delta}\gamma^p_{\delta\beta} + \gamma^{st}_{\beta\delta}\gamma^p_{\delta\alpha}\right)\epsilon_{ABC}q_s{}^B q_t{}^C\nabla_p{}^A \\ &= -\frac{i}{2}\left(\gamma^s_{\alpha\beta}\epsilon_{ABC}q_s{}^B q_t{}^C\nabla_t{}^A - \gamma^t_{\alpha\beta}\epsilon_{ABC}q_s{}^B q_t{}^C\nabla_s{}^A\right) \\ &= -\frac{i}{2}\gamma^t_{\alpha\beta}q_t{}^A\epsilon_{ABC}\left(q_s{}^B\nabla_s{}^C - q_s{}^C\nabla_s{}^B\right). \end{aligned} \tag{4.45}$$

The only terms in III and IV we have not yet dealt with are those without derivatives, namely,

$$-i\Theta^{[B}_{(\delta}\Theta^{A]}_{\gamma)}\left(\gamma^s_{\delta\beta}\gamma^{st}_{\alpha\gamma} + \gamma^s_{\delta\alpha}\gamma^{st}_{\beta\gamma}\right)\epsilon_{ABC}q_t{}^C. \tag{4.46}$$

Since it is not matrix multiplication of the Dirac matrices we are forced to Fierz the expression. We immediately note that due to symmetry requirements and the fact that $\gamma^{(5\leq n\leq 9)}$ is already included in $\gamma^{(0\leq n\leq 4)}$, the only terms that remain are those for $n = 0$ and $n = 4$,

$$\begin{aligned} \gamma^s_{(\alpha(\gamma}\gamma^{st}_{\beta)\delta)} &= a\,\delta_{\alpha\beta}\gamma^t_{\gamma\delta} + b\,\gamma^t_{\alpha\beta}\delta_{\gamma\delta} \\ &\quad + c\,\gamma^{s_1\ldots s_4}_{\alpha\beta}\gamma^{s_1\ldots s_4 t}_{\gamma\delta} + d\,\gamma^{s_1\ldots s_4 t}_{\alpha\beta}\gamma^{s_1\ldots s_4}_{\gamma\delta}. \end{aligned} \tag{4.47}$$

By using the totally antisymmetric tensor on the last term we can transform it into the c-term. We thus exclude the last term, and then contract both sides of the equation with $(\gamma^t)^{\beta\gamma}$. This yields,

$$\begin{aligned} L.H.S &= \frac{1}{4}\left(\gamma^t\right)^{\beta\gamma}\left(\gamma^s_{\alpha\gamma}\gamma^{st}_{\beta\delta} + \gamma^s_{\alpha\delta}\gamma^{st}_{\beta\gamma} + \gamma^s_{\beta\gamma}\gamma^{st}_{\alpha\delta} + \gamma^s_{\beta\delta}\gamma^{st}_{\alpha\gamma}\right) \\ &= \frac{1}{4}\left(\left(\gamma^s\gamma^t\gamma^{st}\right)_{\alpha\delta} - \left(\gamma^s\gamma^t\gamma^{st}\right)_{\alpha\delta}\right) = 0. \end{aligned} \tag{4.48}$$

The contracted right-hand side of (4.47) becomes,

$$\begin{aligned} R.H.S &= a\left(\gamma^t\gamma^t\right)_{\alpha\delta} + b\left(\gamma^t\gamma^t\right)_{\alpha\delta} + 5c\gamma^{s_1\ldots s_4}_{\alpha\beta}\gamma^{s_1\ldots s_4}_{\gamma\delta} \\ &= 9a\delta_{\alpha\delta} + 9b\delta_{\alpha\delta} + 5(-6)7(-8)9c\delta_{\alpha\delta}. \end{aligned} \tag{4.49}$$

Thus we have the relation,

$$a + b + 1680c = 0. \tag{4.50}$$

To proceed, we now contract (4.47) with $\delta^{\alpha\beta}$, giving us

$$L.H.S \;=\; \frac{1}{4}\left(\gamma^s\gamma^{st} + \gamma^s\gamma^{st} + \gamma^s\gamma^{st} + \gamma^s\gamma^{st}\right) = 8\gamma^t, \tag{4.51}$$

and

$$R.H.S \;=\; 16a\gamma^t. \tag{4.52}$$

Instead contracting with $\delta^{\gamma\delta}$ produces a similar relation for the b-term, except for an additional minus sign hailing from the antisymmetry of γ^{st}. We thus have the following:

$$a = \frac{1}{2}, \qquad b = -\frac{1}{2}. \tag{4.53}$$

Putting this result into (4.50) gives us $c = 0$, and we can write down the Fierzed equation,

$$\gamma^s_{\delta\beta}\gamma^{st}_{\alpha\gamma} + \gamma^s_{\delta\alpha}\gamma^{st}_{\beta\gamma} = \delta_{\alpha\beta}\gamma^t_{\gamma\delta} - \gamma^t_{\alpha\beta}\delta_{\gamma\delta}, \tag{4.54}$$

which, inserted into (4.46), becomes

$$-i\delta_{\alpha\beta}q_t{}^C\epsilon_{ABC}\Theta^{[B}_{(\delta}\Theta^{A]}_{\gamma)}\gamma^t_{\gamma\delta} + i\gamma^t_{\alpha\beta}q_t{}^C\epsilon_{ABC}\Theta^{[B}_{\gamma}\Theta^{A]}_{\gamma}. \tag{4.55}$$

This we can rewrite as

$$i\delta_{\alpha\beta}\vec{q}_s \cdot \left(\vec{\Theta}_\gamma \times \vec{\Theta}_\delta\right)\gamma^s_{\gamma\delta} - \frac{1}{2}i\gamma^t_{\alpha\beta}q_t{}^A\left(\Theta^B_\gamma\Theta^C_\gamma - \Theta^C_\gamma\Theta^B_\gamma\right). \tag{4.56}$$

We have now finished the calculation of the anticommutation relations of the supercharges. Gathering the results we have,

$$\{Q_\alpha, Q_\beta\} = \delta_{\alpha\beta}H + \gamma^t_{\alpha\beta}q_{tA}\epsilon_{ABC}J_{BC}, \tag{4.57}$$

with the Hamiltonian being

$$H = -\sum_{s=1}^{9}\vec{\nabla}^2_s + \sum_{s<t}(\vec{q}_s \times \vec{q}_t)^2 + i\vec{q}_s \cdot (\vec{\Theta}_\alpha \times \vec{\Theta}_\beta)\gamma^s_{\alpha\beta}, \tag{4.58}$$

which furthermore commutes with J_{AB} and J_{st}.

4.2.2 The theorem

The question we want answered is now whether there exist a normalizable state $\Psi \in \mathcal{H}$, that is a singlet with respect to $SU(2)$ and $Spin(d)$, and fulfills

$$H\psi = 0 \tag{4.59}$$

i.e., have zero energy. An equivalent, but much easier route is to look for the zero-modes,

$$Q_\beta \psi = 0, \qquad \beta = 1, \ldots, s_d. \tag{4.60}$$

It should also be noted that the requirement of $Spin(d)$ invariance is not trivial. The full argument is presented in [61].

Introducing the variable $r > 0$, and the unit vectors $\vec{e}$ and E_s obeying

$$\vec{e}^{\,2} = 1, \qquad \sum_s E_s^2 = 1, \tag{4.61}$$

the bosonic potential $\sum_{s<t}(\vec{q}_s \times \vec{q}_t)^2$ will vanish on the manifold

$$\vec{q}_s = r\vec{e}E_s. \tag{4.62}$$

The dimension of this manifold is

$$\underbrace{1}_{r} + \underbrace{2}_{\vec{e}} + \underbrace{(d-1)}_{E_s}. \tag{4.63}$$

Furthermore, we can express points in a conical neighborhood of the manifold by making use of tubular coordinates,

$$\vec{q}_s = r\vec{e}E_s + r^{-1/2}\vec{y}_s, \tag{4.64}$$

where $r^{-1/2}$ have been added not out of necessity but to simplify for us later on. The transversal coordinates $\vec{y}_s$ obey

$$\vec{y}_s \cdot \vec{e} = 0_s, \qquad \vec{y}_s E_s = \vec{0}. \tag{4.65}$$

We also need to remark that $\vec{q}_s$ is invariant under the antipode map,

$$(\vec{e}, E, y) \longrightarrow (-\vec{e}, -E, y). \tag{4.66}$$

Therefore we will look for, and include, only those states that are even under this transformation.

We are now ready to present the theorem describing the possible ground state. The formulation of the theorem is taken verbatim from [10].

Theorem 4.1 *Consider the equations $Q_\beta \psi = 0$ for a formal power series solution near $r = \infty$ of the form*

$$\psi = r^{-\kappa} \sum_{k=0}^{\infty} r^{-\frac{3}{2}k} \psi_k, \tag{4.67}$$

where : $\psi_k = \psi_k(\vec{e}, E, y)$ *is square-integrable w.r.t.* $\mathrm{d}e\, \mathrm{d}E\, \mathrm{d}y$;
ψ_k *is* $\mathrm{SU}(2) \times \mathrm{Spin}(d)$ *invariant*;
$\psi_0 \neq 0$.

Then, up to linear combinations,

- $d = 9$*: The solution is unique, and* $\kappa = 6$*;*
- $d = 5$*: There are three solutions with* $\kappa = -1$ *and one with* $\kappa = 3$*;*
- $d = 3$*: There are two solutions with* $\kappa = 0$*;*
- $d = 2$*: There are no solutions.*

All solutions are even under the antipode map (4.66),

$$\psi_k(\vec{e}, E, y) = \psi_k(-\vec{e}, -E, y), \tag{4.68}$$

except for the state $d = 5$, $\kappa = 3$, *which is odd, and thus not a viable ground state.*

When we check whether a possible ground state is square-integrable or not we use the integration measure

$$dq = dr \cdot r^2 de \cdot r^{d-1} dE \cdot r^{\frac{1}{2}\cdot 2(d-1)} dy = r^2 dr\, de\, dE\, dy \tag{4.69}$$

Thus, for ψ to be square integrable (at infinity)

$$\int^{\infty} dr\, r^2 (r^{-\kappa})^2 < \infty \tag{4.70}$$

must hold. This is fulfilled if $\kappa > 3/2$. Hence we can immediately remove the solutions for $d = 3$, and $d = 5$ with $\kappa = -1$. As was already pointed out, the solution $d = 5$ with $\kappa = 3$ is odd under the antipode map (4.66) and thus cannot be a ground state. This makes $d = 9$ the sole survivor, and as this case corresponds to the eleven-dimensional supermembrane, it makes us warm all over.

4.3 The Proof

To prove theorem 4.1 we will expand the supercharges in power series and match these with the power series of the conjectured wave functions (4.67). This will yield $n \to \infty$ equations, where $n = 0$ corresponds to $Q^0_\beta \psi_0 = 0$. We then proceed by solving this initial case, and find a solution that is not necessarily $SU(2) \times Spin(d)$ invariant. Next we check what states are singlets of $SU(2) \times Spin(d)$ and proceed to check that the states are even under the antipode map (4.66). We then continue by investigating the equations for $n > 0$ and then finally determine the different values of κ for the various values of d.

4.3.1 The $d = 2$ case

The fact that no ground state can exist for $d = 2$ can easily be derived without going through the full proof. By examining J_{st} and showing that no state J_{12} satisfying

$$J_{12}\psi = 0 \tag{4.71}$$

exists, it follows that no $Spin(d)$ invariant ground state is possible for $d = 2$. We start by constructing M_{12}. For $d = 2$ we have $s_d = 2$ and can use the standard Pauli matrices, choosing

$$\gamma^1 = \sigma^1, \qquad \gamma^2 = \sigma^3. \tag{4.72}$$

Thus yielding,

$$\gamma^{12} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{4.73}$$

For M_{12} we now get,

$$\begin{aligned} M_{12} &= -\frac{i}{4}\Theta_{\alpha A}\gamma^{12}_{\alpha\beta}\Theta_{\beta A} = \frac{i}{4}\left(\Theta_{1A}\Theta_{2A} - \Theta_{2A}\Theta_{1A}\right) \\ &= \frac{i}{2}\Theta_{1A}\Theta_{2A}, \end{aligned} \tag{4.74}$$

where we sum over $A = 1, 2, 3$. We will also need the six Clifford generators $\Theta_{\alpha A}$. We construct these according to[2],

$$\begin{aligned} \Theta_{\alpha 1} &= \frac{1}{\sqrt{2}}\left(\sigma_\alpha \otimes \mathbb{1}_2 \otimes \sigma_3\right) \\ \Theta_{\alpha 2} &= \frac{1}{\sqrt{2}}\left(\sigma_3 \otimes \sigma_\alpha \otimes \mathbb{1}_2\right) \\ \Theta_{\alpha 3} &= \frac{1}{\sqrt{2}}\left(\mathbb{1}_2 \otimes \sigma_3 \otimes \sigma_\alpha\right). \end{aligned}$$

[2]More on this scheme can be found in appendix C.

Explicitly using these 8×8 matrices and forming $(M_{12})^2$ we obtain a diagonal matrix whose elements are the squared eigenvalues of M_{12}. The extracted eigenvalues are $1/4$ and $3/4$. The final observation that need to be done is that L_{12} has the spectrum $\mathbb{Z}$, thus making any state fulfilling $J_{12}\psi = 0$ impossible.

4.3.2 Power series expansion of Q_β

To make a power series expansion of the supercharges Q_β we first need to obtain the derivative

$$\begin{aligned}\frac{\partial}{\partial q_{tA}} &= r^{1/2}(\delta_{st} - E_s E_t)(\delta_{AB} - e_A e_B)\frac{\partial}{\partial y_{sB}} \\ &+r^{-1}\left(e_A E_t\left(r\frac{\partial}{\partial r} + \frac{1}{2}y_{sB}\frac{\partial}{\partial y_{sB}}\right) + ie_B E_t L_{BA} + ie_A E_s L_{st}\right) \\ &+\mathcal{O}(r^{-5/2}). \end{aligned} \tag{4.75}$$

To show this we need to calculate the partial derivatives contained in

$$\frac{\partial}{\partial q_{tA}} = \frac{\partial r}{\partial q_{tA}}\frac{\partial}{\partial r} + \frac{\partial e_B}{\partial q_{tA}}\frac{\partial}{\partial e_B} + \frac{\partial E_s}{\partial q_{tA}}\frac{\partial}{\partial E_s} + \frac{\partial y_{sB}}{\partial q_{tA}}\frac{\partial}{\partial y_{sB}}. \tag{4.76}$$

From

$$\vec{q}_s = r\vec{e}E_s + r^{-1/2}\vec{y}_s \tag{4.77}$$

we get

$$dq_{tA} = \left(e_A E_t - \frac{1}{2}r^{-3/2}y_{tA}\right)dr + rE_t de_A + re_A dE_t + r^{-1/2}dy_{tA}. \tag{4.78}$$

We then use the coordinate relations (4.65) and (4.61) to obtain

$$e_A dy_{tA} + y_{tA} de_A = 0 \tag{4.79}$$

$$E_t dy_{tA} + y_{tA} dE_t = 0 \tag{4.80}$$

$$e_A de_A = 0 \tag{4.81}$$

$$E_t dE_t = 0. \tag{4.82}$$

Using these relations we derive the following contractions of (4.78),

$$e_A E_t dq_{tA} = dr, \tag{4.83}$$

$$(\delta_{AB} - e_A e_B)E_t dq_{tA} = rde_B - r^{-1/2}y_{tB}dE_t, \tag{4.84}$$

$$e_A(\delta_{st} - E_s E_t)dq_{tA} = rdE_s - r^{-1/2}y_{sA}de_A, \tag{4.85}$$

$$\begin{aligned}(\delta_{AB} - e_A e_B)(\delta_{st} - E_s E_t)dq_{tA} &= -\frac{1}{2}r^{-1/2}(dy_{sB} + e_B y_{sA} de_A \\ &+E_s y_{tB} dE_t).\end{aligned} \tag{4.86}$$

To proceed we introduce two matrices m and M:

$$m_{AB} = \delta_{AB} - r^{-3} y_{tA} y_{tB}, \tag{4.87}$$
$$M_{st} = \delta_{st} - r^{-3} y_{sA} y_{tA}. \tag{4.88}$$

These matrices are then used to express the differentials de_B and dE_s:

$$de_B = (m^{-1})_{BC}(r^{-1}(\delta_{CA} - e_C e_A)E_t + r^{-5/2} y_{tC} e_A) dq_{tA}, \tag{4.89}$$
$$dE_s = (M^{-1})_{su}(r^{-1}(\delta_{ut} - E_u E_t)e_A + r^{-5/2} y_{sA} E_t) dq_{tA}. \tag{4.90}$$

The validity of the above statements is most easily checked by multiplying equation (4.84) with both m and M, and then inserting the expressions (4.87)-(4.90) and checking that the equality still holds. By expanding m and M we rewrite the differentials as

$$de_B = (r^{-1}(\delta_{BA} - e_B e_A)E_t + \mathcal{O}(r^{-5/2}))dq_{tA}, \tag{4.91}$$
$$dE_s = (r^{-1}(\delta_{st} - E_s E_t)e_A + \mathcal{O}(r^{-5/2}))dq_{tA}. \tag{4.92}$$

We can now write down the four differentials

$$dr = e_A E_t dq_{tA}, \tag{4.93}$$
$$de_B = (r^{-1}(\delta_{BA} - e_B e_A)E_t + \mathcal{O}(r^{-5/2}))dq_{tA}, \tag{4.94}$$
$$dE_s = (r^{-1}(\delta_{st} - E_s E_t)e_A + \mathcal{O}(r^{-5/2}))dq_{tA}, \tag{4.95}$$
$$\begin{aligned} dy_{sB} = {} & [r^{1/2}(\delta_{BA} - e_B e_A)(\delta_{st} - E_s E_t) \\ & + \frac{1}{2} r^{-1} e_A E_t y_{sB}] dq_{tA} - e_B y_{sA} de_A - E_s y_{tB} dE_t, \end{aligned} \tag{4.96}$$

which makes it a simple matter to express the partial derivatives in (4.76) as:

$$\begin{aligned} \frac{\partial}{\partial q_{tA}} = {} & r^{1/2}(\delta_{st} - E_s E_t)(\delta_{AB} - e_A e_B)\frac{\partial}{\partial y_{sB}} \\ & + r^{-1}[e_A E_t (r \det r + \frac{1}{2} y_{sB} \frac{\partial}{\partial y_{sB}})] \\ & + r^{-1}(\delta_{AC} - e_A e_C)E_t(\delta_{CB}\frac{\partial}{\partial e_B} - e_B y_{sC}\frac{\partial}{\partial y_{sB}}) \\ & + r^{-1}(\delta_{ut} - E_u E_t)e_A(\delta_{us}\frac{\partial}{\partial E_s} - E_s y_{uB}\frac{\partial}{\partial y_{sB}}) \\ & + \mathcal{O}(r^{-5/2}), \end{aligned} \tag{4.97}$$

where the last term does not contain any derivatives with respect to r. By inserting this into iL_{BA} we get

$$\begin{aligned} iL_{BA} &= q_{sB}\frac{\partial}{\partial q_{sA}} - q_{sA}\frac{\partial}{\partial q_{sB}} \\ &= [(\delta_{AC} - e_A e_C)y_{sB} - (\delta_{BC} - e_B e_C)y_{sA}]\frac{\partial}{\partial y_{sB}} \\ &\quad + e_B(\delta_{AC}\frac{\partial}{\partial e_C} - e_C y_{sA}\frac{\partial}{\partial y_{sC}}) \\ &\quad - e_A(\delta_{BC}\frac{\partial}{\partial e_C} - e_C y_{sB}\frac{\partial}{\partial y_{sC}}). \end{aligned} \tag{4.98}$$

This allows us to write down

$$ir^{-1}e_B E_t L_{BA} = r^{-1}(\delta_{AC} - e_A e_C)E_t(\delta_{CB}\frac{\partial}{\partial e_B} - e_B y_{sC}\frac{\partial}{\partial y_{sB}}), \tag{4.99}$$

and in analogy with this, also

$$ir^{-1}e_A E_s L_{st} = r^{-1}(\delta_{ut} - E_u E_t)e_A(\delta_{us}\frac{\partial}{\partial E_s} - E_s y_{uB}\frac{\partial}{\partial y_{sB}}), \tag{4.100}$$

which, together with (4.97) forms our sought-after derivative (4.75).

We are now armed and ready to expand Q_β,

$$\begin{aligned} Q_\beta &= \vec{\Theta}_\alpha \cdot (-i\gamma^t_{\alpha\beta}\vec{\nabla}_t + \frac{1}{2}\vec{q}_s \times \vec{q}_t \gamma^{st}_{\beta\alpha}) \\ &= -i\Theta_{\alpha A}\gamma^t_{\alpha\beta} r^{1/2}(\delta_{st} - E_s E_t)(\delta_{AB} - e_A e_B)\frac{\partial}{\partial y_{sB}} \\ &\quad - i\Theta_{\alpha A}\gamma^t_{\alpha\beta} r^{-1}(e_A E_t r(\frac{\partial}{\partial t} + \frac{1}{2}y_{sB}\frac{\partial}{\partial y_{sB}})) \\ &\quad + \Theta_{\alpha A}\gamma^t_{\alpha\beta} r^{-1}(e_B E_t L_{BA} + e_A E_s L_{st}) \\ &\quad + \vec{\Theta}_\alpha \cdot r^{1/2}(\vec{e} \times \vec{y}_t)E_s \gamma^{st}_{\beta\alpha} \\ &\quad + \frac{1}{2}\vec{\Theta}_\alpha \cdot r^{-1}(\vec{y}_s \times \vec{y}_t)\gamma^{st}_{\beta\alpha} + \mathcal{O}(r^{-5/2}). \end{aligned} \tag{4.101}$$

Next we identify the r-independent operators which we will match with orders of $r^{-\frac{3n}{2}}$,

$$\begin{aligned} Q^0_\beta &= -i\Theta_{\alpha A}\gamma^t_{\alpha\beta}(\delta_{st} - E_s E_t)(\delta_{AB} - e_A e_B)\frac{\partial}{\partial y_{sB}} \\ &\quad + \vec{\Theta}_\alpha \cdot (\vec{e} \times \vec{y}_t)E_s \gamma^{st}_{\beta\alpha}, \end{aligned} \tag{4.102}$$

$$\hat{Q}^1_\beta = -i(\vec{\Theta}_\alpha \cdot \vec{e})\gamma^t_{\alpha\beta}E_t, \tag{4.103}$$

$$\begin{aligned} Q^1_\beta &= \Theta_{\alpha A}\gamma^t_{\alpha\beta}(e_B E_t L_{BA} + e_A E_s L_{st} - \frac{i}{2}e_A E_t y_{sB}\frac{\partial}{\partial y_{sB}}) \\ &\quad + \frac{1}{2}\vec{\Theta}_\alpha(\vec{y}_s \times \vec{y}_t)\gamma^{st}_{\beta\alpha}. \end{aligned} \tag{4.104}$$

Higher order terms will not be needed explicitly. We now pair these operators with powers of r and get the power series expansion of Q_β:

$$Q_\beta = \underbrace{r^{1/2}Q^0_\beta}_{\bar{Q}^0_\beta} + \underbrace{r^{-1}(\hat{Q}^1_\beta r\frac{\partial}{\partial r} + Q^1_\beta)}_{\bar{Q}^1_\beta} + \underbrace{r^{-5/2}Q^2_\beta}_{\bar{Q}^2_\beta} + \dots . \quad (4.105)$$

Now, before we write down the equation $Q_\beta\psi = 0$ in all its glory we need to introduce an alternative notation for ψ (cf. previous definition (4.67)),

$$\psi = \sum_n^\infty \bar{\psi}_n. \quad (4.106)$$

By noting the powers of r in $\bar{Q}^n_\beta$ and $\bar{\psi}_n$,

$$\bar{Q}^m_\beta \sim r^{\frac{1}{2}-\frac{3m}{2}}, \quad (4.107)$$

$$\bar{\psi}_n \sim r^{-(\kappa+\frac{3n}{2})}, \quad (4.108)$$

we find the m-independent quantity,

$$\bar{Q}^m_\beta\bar{\psi}_{n-m} \sim r^{-\kappa-\frac{3n}{2}+\frac{1}{2}}. \quad (4.109)$$

Thus we can finally write down the equation $Q_\beta\psi = 0$,

$$Q^0_\beta\psi_n + (-(\kappa + \frac{3}{2}(n-1))\hat{Q}^1_\beta + Q^1_\beta)\psi_{n-1} + Q^2_\beta\psi_{n-2} + \dots + Q^n_\beta\psi_0 = 0, \quad (4.110)$$

where $n = 0, 1, \dots$.

4.3.3 The equation at $n = 0$

For $n = 0$ the equation (4.110) is simply

$$Q^0_\beta\psi_0 = 0. \quad (4.111)$$

This equation admits precisely the solutions

$$\psi_0(\vec{e}, E, y) = e^{-\sum_s y_s^2/2} |F(E, \vec{e})\rangle . \quad (4.112)$$

This solution, however, is not necessarily $SU(2)\times Spin(d)$ invariant, and imposing this condition is something we will do later. To describe the fermionic states $|F(E,\vec{e})\rangle$ we first introduce the two complex vectors $\vec{n}_\pm$, satisfying

$$\vec{n}_+ \cdot \vec{n}_- = 1 \quad (4.113)$$

$$\vec{e} \times \vec{n}_\pm = \mp i\vec{n}_\pm, \quad (4.114)$$

and thus also

$$\begin{aligned} \vec{n}_\pm \cdot \vec{n}_\pm &= 0 & (4.115)\\ \vec{n}_+ \times \vec{n}_- &= -i\vec{e}. & (4.116) \end{aligned}$$

Explicit derivations of $\vec{n}_\pm$ are done in appendix C. We now introduce the vectors $u, v \in \mathbb{R}^{s_d}$ and $\vec{\Theta}(v) = \vec{\Theta}_\alpha v_\alpha$, allowing us to construct the fermionic operators

$$\vec{\Theta}(v) \cdot \vec{n}_\pm. \tag{4.117}$$

These operators satisfy the anticommutation relations,

$$\begin{aligned} \{\vec{\Theta}(u) \cdot \vec{n}_+, \vec{\Theta}(v) \cdot \vec{n}_-\} &= \{u_\alpha \Theta_{\alpha A} n_+^A, v_\beta \Theta_{\beta B} n_-^B\} \\ &= u_\alpha v_\beta n_+^A n_-^B \delta_{\alpha\beta}\delta_{AB} \\ &= u_\alpha v_\alpha (\vec{n}_+ \cdot \vec{n}_-) \\ &= u_\alpha v_\alpha. \end{aligned} \tag{4.118}$$

By the same procedure and using (4.115) we also obtain

$$\{\vec{\Theta}(u) \cdot \vec{n}_\pm, \vec{\Theta}(v) \cdot \vec{n}_\pm\} = 0. \tag{4.119}$$

Next, we want to calculate $\{Q_\alpha^0, Q_\beta^0\}$, and as we have already done this calculation for the complete supercharges we only give the results,

$$\{Q_\alpha^0, Q_\beta^0\} = \delta_{\alpha\beta} H^0 + \gamma_{\alpha\beta}^t E_t \epsilon_{ABC} M_{AB} e_C, \tag{4.120}$$

where the Hamiltonian is,

$$\begin{aligned} H^0 &= \left(-(\delta_{st} - E_s E_t)(\delta_{AB} - e_A e_B) \frac{\partial}{\partial y_{sA}} \frac{\partial}{\partial y_{tB}} + \sum_s \vec{y}_s^2 \right) \\ &\quad + i\, E_s \gamma_{\alpha\beta}^s \vec{e} \cdot (\vec{\Theta}_\alpha \times \vec{\Theta}_\beta) = H_B^0 + H_F^0. \end{aligned} \tag{4.121}$$

What we now aim to do is prove that the fermionic states $|F(E, \vec{e})\rangle$ have to satisfy

$$\vec{\Theta}(v) \cdot \vec{n}_\pm \, |F(E, \vec{e})\rangle = 0, \quad \text{when} \quad E_s \gamma^s v = \pm v. \tag{4.122}$$

In other words, we want to show that equation (4.111) implies (4.122). To accomplish this we begin by contracting equation (4.120) with $\delta_{\alpha\beta}$ and then with $\gamma_{\alpha\beta}^t E_t$ we see that equation (4.111) in fact correspond to,

$$\begin{aligned} H^0 \psi_0 &= 0 & (4.123)\\ \epsilon_{ABC} M_{AB} e_C \psi_0 &= 0. & (4.124) \end{aligned}$$

The next step is to show that these two equations are satisfied if and only if (4.122) holds. We begin by noting that the bosonic Hamiltonian, H_B^0, is a harmonic oscillator with $2(d-1)$ degrees of freedom. Moreover, it carries the energy $2(d-1)$ and as a ground state have the wave function,

$$\varphi_B^0 = e^{-\sum_s \frac{\vec{y}_s^2}{2}}. \tag{4.125}$$

To investigate the fermionic part of the Hamiltonian, we write

$$\begin{aligned} H_F^0 &= iE_s\gamma^s_{\alpha\beta}\vec{e}\cdot(\vec{\Theta}_\alpha \times \vec{\Theta}_\beta) = -E_s\gamma^s_{\alpha\beta}(\vec{n}_+ \times \vec{n}_-)(\vec{\Theta}_\alpha \times \vec{\Theta}_\beta) \\ &= E_s\gamma^s_{\alpha\beta}\vec{\Theta}_\alpha \cdot (\vec{n}_+ \times \vec{n}_-) \times \vec{\Theta}_\beta = -E_s\gamma^s_{\alpha\beta}\vec{\Theta}_\alpha\vec{\Theta}_\beta \times (\vec{n}_+ \times \vec{n}_-) \\ &= -E_s\gamma^s_{\alpha\beta}((\vec{\Theta}_\alpha \cdot \vec{n}_+)(\vec{\Theta}_\beta \cdot \vec{n}_-) - (\vec{\Theta}_\alpha \cdot \vec{n}_-)(\vec{\Theta}_\beta \cdot \vec{n}_+)). \end{aligned} \tag{4.126}$$

To proceed further we decompose $E_s\gamma^s_{\alpha\beta}$ into projection operators,

$$E_s\gamma^s_{\alpha\beta} = P^+ - P^-, \tag{4.127}$$

where the $s_d \times s_d$ matrices have half of their diagonal elements set to unity,

$$P^+ = \mathrm{diag}(1,\ldots,1,0,\ldots,0), \;\; P^- = \mathrm{diag}(0,\ldots,0,1,\ldots,1). \tag{4.128}$$

To continue we now get,

$$\begin{aligned} (4.126) &= -P^+_{\alpha\beta}(\vec{\Theta}_\alpha \cdot \vec{n}_+)(\vec{\Theta}_\beta \cdot \vec{n}_-) + P^+_{\alpha\beta}(\vec{\Theta}_\alpha \cdot \vec{n}_-)(\vec{\Theta}_\beta \cdot \vec{n}_+) \\ &\quad + P^-_{\alpha\beta}(\vec{\Theta}_\alpha \cdot \vec{n}_+)(\vec{\Theta}_\beta \cdot \vec{n}_-) - P^-_{\alpha\beta}(\vec{\Theta}_\alpha \cdot \vec{n}_-)(\vec{\Theta}_\beta \cdot \vec{n}_+) \\ &= -P^+_{\alpha\alpha} + P^+_{\alpha\beta}(\vec{\Theta}_\alpha \cdot \vec{n}_-)(\vec{\Theta}_\beta \cdot \vec{n}_+) + P^+_{\alpha\beta}(\vec{\Theta}_\alpha \cdot \vec{n}_-)(\vec{\Theta}_\beta \cdot \vec{n}_+) \\ &\quad + P^-_{\alpha\beta}(\vec{\Theta}_\alpha \cdot \vec{n}_+)(\vec{\Theta}_\beta \cdot \vec{n}_-) - P^-_{\alpha\alpha} + P^-_{\alpha\beta}(\vec{\Theta}_\alpha \cdot \vec{n}_+)(\vec{\Theta}_\beta \cdot \vec{n}_-) \\ &= -s_d + 2P^+_{\alpha\beta}(\vec{\Theta}_\alpha \cdot \vec{n}_-)(\vec{\Theta}_\beta \cdot \vec{n}_+) + 2P^-_{\alpha\beta}(\vec{\Theta}_\alpha \cdot \vec{n}_+)(\vec{\Theta}_\beta \cdot \vec{n}_-) \end{aligned} \tag{4.129}$$

where we used the Clifford algebra and the structure of the projection operators. The attentive reader immediately note from (4.25) that

$$s_d = 2(d-1). \tag{4.130}$$

Thus will the first term in H_F^0 cancel the energy contribution from the bosonic part of the Hamiltonian. Furthermore we see that the remaining terms in H_F^0 kill the state $|F(E,\vec{e})\rangle$ if the condition (4.122) holds. In other words, $H^0\psi_0 = 0$, and we are left to show that (4.124) annihilates

$|F(E,\vec{e})\rangle$,

$$\begin{aligned}
\epsilon_{ABC}M_{AB}e_C\psi_0 &= -\frac{i}{2}\epsilon_{ABC}(\Theta_{\alpha A}\Theta_{\alpha B}-\Theta_{\alpha B}\Theta_{\alpha A})e_C \\
&= -i\epsilon_{ABC}\Theta_{\alpha A}\Theta_{\alpha B}e_C \\
&= -i\vec{e}\cdot(\vec{\Theta}_\alpha\times\vec{\Theta}_\alpha) \\
&= (\vec{n}_+\times\vec{n}_-)\cdot(\vec{\Theta}_\alpha\times\vec{\Theta}_\alpha) \\
&= (\vec{\Theta}_\alpha\cdot\vec{n}_+)(\vec{\Theta}_\alpha\cdot\vec{n}_-)-(\vec{\Theta}_\alpha\cdot\vec{n}_-)(\vec{\Theta}_\alpha\cdot\vec{n}_+), \quad (4.131)
\end{aligned}$$

where we in the second to last equality used the same reasoning as in the previous calculation for H_F^0. Now using the structure of the projection operators, we get

$$\begin{aligned}
\ldots &= P^+_{\alpha\beta}(\vec{\Theta}_\alpha\cdot\vec{n}_+)(\vec{\Theta}_\alpha\cdot\vec{n}_-)+P^-_{\alpha\beta}(\vec{\Theta}_\alpha\cdot\vec{n}_+)(\vec{\Theta}_\alpha\cdot\vec{n}_-) \\
&\quad -P^+_{\alpha\beta}(\vec{\Theta}_\alpha\cdot\vec{n}_-)(\vec{\Theta}_\alpha\cdot\vec{n}_+)-P^+_{\alpha\beta}(\vec{\Theta}_\alpha\cdot\vec{n}_-)(\vec{\Theta}_\alpha\cdot\vec{n}_+) \\
&= 2P^-_{\alpha\beta}(\vec{\Theta}_\alpha\cdot\vec{n}_+)(\vec{\Theta}_\alpha\cdot\vec{n}_-)-2P^+_{\alpha\beta}(\vec{\Theta}_\alpha\cdot\vec{n}_-)(\vec{\Theta}_\alpha\cdot\vec{n}_+), \quad (4.132)
\end{aligned}$$

where we in the last equality used the Clifford algebra. It is immediately apparent that this expression annihilates $|F(E,\vec{e})\rangle$.

To recapitulate, we have thus derived the form (4.112) of the solutions to the equation $Q^0_\beta\psi_0=0$, and specifically shown the conditions (4.122) the state $|F(E,\vec{e})\rangle$ must obey.

4.3.4 Extracting allowed states

The remaining parts of the proof will not be given in full detail. Instead the remaining steps will be given a slightly more brief explanation. The reader who wish to delve into the details is referred to [10].

Having obtained and described ψ_0 satisfying $Q^0_\beta\psi_0=0$, the next step is to extract the states which are $SU(2)\times Spin(d)$ invariant. As we mentioned in section 4.2.1, we have $Spin(d)\hookrightarrow SO(s_d)$. Letting $R\in SO(s_d)$ and $U\in SU(2)$ there is a natural representation of $SU(2)\times Spin(d)$ on the Hilbert space $\mathcal{H}=\mathrm{L}^2(X,\mathcal{C}^{\otimes 3})$. $SU(2)\times Spin(d)$ act naturally on X through the representation $SO(3)\times SO(d)$. In the case of $\mathcal{C}^{\otimes 3}$ we have the representation $\mathcal{R}$ of $Spin(s_d)\ni R$, satisfying

$$\mathcal{R}(R)^*\Theta_{\alpha A}\mathcal{R}(R)=\tilde{R}_{\alpha\beta}\Theta_{\beta A}, \quad (4.133)$$

where $\tilde{R}=\tilde{R}(R)$ is its above mentioned $SO(s_d)$ representation. Using (4.26) together with the fact that

$$SO(s_d)=Spin(s_d)/(Z)_2, \quad (4.134)$$

we obtain

$$Spin(d) \hookrightarrow Spin(s_d). \tag{4.135}$$

In other words, we have shown that $\mathcal{R}$ is a representation of $Spin(d)$. Similarly, we have the representation $\mathcal{U}$ of $SU(2) \ni U$ on $\mathcal{C}^{\otimes 3}$,

$$\mathcal{U}(U)^* \Theta_{\alpha A} \mathcal{U}(U) = U_{AB} \Theta_{\alpha B}. \tag{4.136}$$

Now, the $SU(2) \times Spin(d)$ invariant states are those satisfying,

$$\mathcal{U}(U) \left| F(E, \vec{e}) \right\rangle = \left| F(E, U\vec{e}) \right\rangle \tag{4.137}$$

$$\mathcal{R}(R) \left| F(E, \vec{e}) \right\rangle = \left| F(RE, \vec{e}) \right\rangle, \tag{4.138}$$

where $(U, R) \in SU(2) \times Spin(d)$. The above states are furthermore in bijective correspondence to the states invariant under $(U, R) \in U(1) \times Spin(d-1)$. In other words states satisfying

$$\mathcal{U}(U) \left| F(E, \vec{e}) \right\rangle = \left| F(E, \vec{e}) \right\rangle \tag{4.139}$$

$$\mathcal{R}(R) \left| F(E, \vec{e}) \right\rangle = \left| F(E, \vec{e}) \right\rangle, \tag{4.140}$$

for some arbitrary but fixed $(E, \vec{e})$, and where $U\vec{e} = \vec{e}$ and $RE = E$.

After replacing $\Theta_{\alpha A}$ with operators $\vec{\Theta}_\alpha \cdot \vec{e}$ which leave the space (4.122) invariant, it turns out that this representation decomposes into

$$\mathcal{C} = \left(2^{(s_d/2)-1}\right)_+ \oplus \left(2^{(s_d/2)-1}\right)_- . \tag{4.141}$$

The embedding (4.135) and the branching of the representation depend on our choice of Dirac matrices. After a certain choice (see [10] for details) the representation of $Spin(d)$ branches into,

$$\mathcal{C} = \begin{cases} (44 \oplus 84) \oplus 128 & , d = 9 \\ (5 \oplus 1 \oplus 1 \oplus 1) \oplus (4 \oplus 4) & , d = 5 \\ 2 \oplus (1 \oplus 1) & , d = 3. \end{cases} \tag{4.142}$$

We will get additional branching for $Spin(d-1) \hookrightarrow Spin(d)$,

$$\mathcal{C} = \begin{cases} (1 \oplus 8_v \oplus 35_v) \oplus (28 \oplus 56_v) \oplus (8_s \oplus 8_c \oplus 56_s \oplus 56_c) & , d-1 = 8 \\ 1 \oplus 1 \oplus 1 \oplus (1 \oplus 4) \oplus (2_+ \oplus 2_-) \oplus (2_+ \oplus 2_-) & , d-1 = 4 \\ (1_1 \oplus 1_{-1}) \oplus 1_0 \oplus 1_0 & , d-1 = 2. \end{cases} \tag{4.143}$$

After some further analysis we find the invariant states of interest, and the $Spin(d)$ representation to which they are associated,

$$\begin{aligned} d = 9: &\quad 44 \\ d = 5: &\quad 1, 1, 1, 5 \\ d = 3: &\quad 1, 1 \end{aligned}$$

The next step in the proof is to check that the states are even under the antipode map (4.66). Doing so excludes only one of the states for $d = 5$. After that we need to analyze the equations (4.110) for $n \geq 1$. Doing this, we obtain relations needed for the final step, determining κ. Accomplishing this task and obtaining the values of κ presented in the formulation of the theorem then concludes the proof.

4.4 A numerical method

An alternate approach to characterize the membrane vacuum state has recently been proposed by Wosiek [63]. The main idea is to solve problems in supersymmetric Yang-Mills quantum mechanics in various dimensions by way of an algebraic program. Wosiek has managed to confirm many old results and also obtain some new ones, especially for the case of $D = 4$ super Yang-Mills theory. The, for our purposes, more interesting case of $D = 10$ (which corresponds the supermembrane in eleven-dimensional spacetime) was investigated further in [64] with some additional results derived in [65].

Supersymmetric Yang-Mills quantum mechanics has many features in common with more advanced field theories and as such provides a nice "theoretician's laboratory" to study supersymmetry in various guises. Beside the importance of $D = 10$ as a model of the supermembrane and its connection to M-theory, this technique have been applied to $SU(2)$ invariant Wess-Zumino quantum mechanics and various dimensions of supersymmetric Yang-Mills quantum mechanics.

The main theme of Wosiek's approach is to implement the Hamiltonian formulation of quantum mechanics into the logical structure of a computer. Vectors in some Hilbert space are represented by lists with dynamically varying size. Quantum operators are then just functions of these lists. The main computational problem is to limit the size of the Hilbert space which of course limits the number of degrees of freedom that can be allowed. This introduced cut-off, however, is easily monitored and by increasing the cut-off one can find convergent results of eigenvalues and thus obtain trustworthy results.

By using the discrete eigenbasis of the occupation number $a^\dagger a$ we have

$$\{|n\rangle\}, \qquad |n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n |0\rangle \,. \tag{4.144}$$

Furthermore, we can write the bosonic coordinate and momentum oper-

Operation	Quantum mechanics	Mathematica
any state	$\lvert st\rangle$	List
sum	$\lvert st_1\rangle + \lvert st_2\rangle$	$\mathrm{Add}[\mathrm{list}_1, \mathrm{list}_2]$
number multiply	$\alpha \lvert st_1\rangle$	$\mathrm{Mult}[\alpha, \mathrm{list}_1]$
scalar product	$\langle st_1 \mid st_2\rangle$	$\mathrm{Sc}[\mathrm{list}_1, \mathrm{list}_2]$
empty state	$\lvert 0\rangle$	$\{1, \{1\}, \{0\}\}$
null state	0	$\{0, \{\}\}$

Table 4.1: Quantum operations and their computer implementation.

ators as

$$x = \frac{1}{\sqrt{2}}(a + a^\dagger) \tag{4.145}$$

$$p = \frac{1}{i\sqrt{2}}(a - a^\dagger). \tag{4.146}$$

Since typical quantum observables are functions of these two operators they can be represented by combinations of the creation and annihilation operators. The fermionic generalization is straightforward but not reviewed here.

A quantum state is a superposition of an arbitrary number, n_s, of elementary states $\lvert n\rangle$, i.e.,

$$\lvert st\rangle = \sum_I^{n_s} a_I \lvert n^{(I)}\rangle . \tag{4.147}$$

In the lingo of Mathematica this would look like

$$st = \{n_s, \{a_1, \ldots, a_{n_s}\}, \{n^{(1)}\}, \{n^{(2)}\}, \ldots, \{n^{(n_s)}\}\}, \tag{4.148}$$

which is a list with $n_s + 2$ elements. The first element in in the list gives the number of elementary states in the linear combination (4.147), the next element is a sublist containing the n_s (real or complex) coefficients a_I, and the remaining n_s elements are all sublists specifying the occupation numbers of elementary states.

To carry out any calculations we must be able to translate quantum mechanical operations into operations on these lists. This turns out to be simple and intuitive, examples of which are shown in table 4.1. Armed with such operations we can construct creation and annihilation operators as list-valued functions. Then we can define the observables we need: supersymmetry generators, Hamiltonians etc.

Consequently, to solve a problem we begin by defining a list corresponding to the empty state, then we proceed by generating a finite basis of $N_{\text{cut-off}}$ vectors and calculate the quantum observables we need. Next, we let the computer execute the program, and if it is the spectrum of the system we are interested in, we simply extract the energy eigenvalues from the Hamiltonian matrix.

The results obtained using this method have been substantial in the case of $D = 4$, which have been fully solved. Specifically, the spectrum have been obtained with its pattern of discrete and continuous states. The cut-off can be chosen high enough for the restrictions on the Hilbert space to become irrelevant.

In $D = 10$ problems arise. The number of states and hence the computational time increase rapidly as we go up in dimensions. No trustworthy results have thus been obtained for the full $D = 10$ case. However, various schemes have been suggested and developed to overcome this problem. One obvious choice is to exclude the fermionic part of the calculation, greatly reducing the number of states involved. By doing this the calculation gets manageable and the lower levels of the spectrum can be found. In the end, the full supersymmetric case is the only case of real interest (at least for our purposes) and judging by the progress made by Wosiek in each consecutive published paper in less than two years optimism seem to be in order. In addition, as computational power is a greatly limiting factor and reading that Wosiek is using a common workstation for his research one cannot help but wonder what results could be obtained if he was given some time on a supercomputer.

5

Conclusions

In our goal to examine the membrane vacuum state, we started out by treating the bosonic membrane and investigating its symmetries, dynamics and quantization properties. After introducing supersymmetry we derived the allowed p-branes and target space dimensionalities. We investigated further the properties of the super p-brane action in the Hamiltonian formulation and the lightcone gauge. The APD algebra and its connection with matrix theory was analyzed and worked through with the example of toroidal membranes.

Dimensional reduction of super Yang-Mills theory was performed and its connection to the description of N D0-branes was commented upon.

The spectrum of both bosonic and supersymmetric membranes was put under close scrutiny and discussed in detail.

In the main chapter we made a brief summary of the research conducted so far in the search for the membrane vacuum state. We then proceeded by investigating a supersymmetric two-dimensional model with an x^2y^2 potential and conclusively proved that no normalizable ground state could exist, a question that remained unsolved for more than a decade and was resolved only recently. We then turned to the central theme of this thesis and treated the $SU(2) \times Spin(d)$ invariant supersymmetric matrix model. In essence, using the first order supercharges Q_β and first order perturbation theory to obtain a solution to

$$Q_\beta \psi = 0, \tag{5.1}$$

where β counts the number of supercharges. We then formulated the theorem regarding the existence and asymptotic form of the possible ground states. For the allowed dimensions $d = 2, 3, 5, 9$ of the model the power law decay of the zero-energy solutions was obtained, with the following result:

- $d = 9$: A unique, normalizable solution.
- $d = 5$: Three solutions, none of which were normalizable.
- $d = 3$: Two solutions, none of which were normalizable.
- $d = 2$: No solutions.

The form of the asymptotic (near $r = \infty$) solution we found for the case of $d = 9$ was,

$$\psi = \sum_{k=0}^{\infty} r^{-\frac{3}{2}k-6}\psi_k, \tag{5.2}$$

where ψ_k is $SU(2) \times Spin(d)$ invariant and square-integrable on surfaces where r is constant.

The dimensionality d of the model is related to the dimensionality D of the spacetime where the supermembrane propagates according to $D = d + 2$. Thus the only normalizable solution correspond to the regulated supermembrane theory in eleven dimensions. Hence the prospect of finding the true membrane vacuum state remain very much a possibility. While work will likely continue on generalizing the $SU(2)$ model to arbitrary N, meanwhile exciting results could very well be obtained from the computational methods of Wosiek, presented in chapter 4.

Having thus briefly summarized what has been done in this thesis, some remarks about what has *not* been done would be in order. In addition, some things which should have been done in greater detail is worthy of comment. The most obvious item would be the slightly abbreviated proof of the main theorem, a proof which certainly deserves to be concluded with the same degree of attention it was begun with.

Another area I would have wished to pursued further is the computational methods of super Yang-Mills quantum mechanics. The method is still in its infancy and I believe it holds great promise in supplying, if nothing else, at least results to serve as a guide for the purely analytical efforts being conducted.

Membrane theory is a surprisingly large field, and though having spent a year within its boundaries there are many aspects that by necessity have been left unstudied. For instance the "spinning membrane" and superembeddings could be argued to have a place in a thesis like this. As would the full proof of the continuity of the quantum supermembrane spectrum.

In the case of the $SU(2)$ matrix model we could have examined further the requirement that the groundstate be $Spin(d)$ invariant, or more thoroughly investigated the work done regarding $SU(N > 2)$ invariant generalizations of the model. This would also be the obvious way to continue the work begun in this thesis. If a generalization could be found to arbitrary N, the membrane vacuum state would be close at hand.

There have also been other attempts at constructing the membrane ground state than those presented here worth mentioning. The main alternative to the approach presented in this thesis is to use Witten indices, as have been done in, e.g., [66, 67]. However, this approach can only hope to prove the *existence* of a vacuum state, not construct it.

In summary, there has been done a great deal toward obtaining the membrane vacuum state, obstacles have been overcome and there are so far no evidence *against* such a state. We thus conclude this thesis by the remark that while there is still room for much work, there is also room for much enthusiasm.

A

Notation and Conventions

In this appendix we try to clarify and explain some of the notation that has been used throughout the thesis. Some quantities will be explained thoroughly while others will simply be listed with a short explanation.

With few exceptions the calculations and results presented in this thesis have been done so in God-given units, i.e., the divine trinity $(\hbar, c, e)$ is set to a Godly unity $(= 1)$. We even take this reductionist scheme a little further by dropping constants such as the membrane tension from most expressions. The reasons for this, however, is related more to laziness than any religious leanings.

A.1 General conventions

- We make use of the so-called "East Coast" metric,

$$\eta^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}. \tag{A.1}$$

- For brevity we will sometimes use the Feynman slash,

$$\not{E}_i = \gamma^\mu E_{\mu i}. \tag{A.2}$$

- The adjoint spinor $\bar{\theta}$ is, as usual,

$$\bar{\theta} = \theta^\dagger \gamma^0. \tag{A.3}$$

Since we limit ourselves to the case where θ is real Γ^0 is nothing but the conjugation matrix C, and thus we have

$$\bar{\theta} = \theta^T C. \tag{A.4}$$

- The conjugation matrix acts as a kind of metric for spinors, e.g.,

$$\theta^\alpha = C^{\alpha\beta}\theta_\beta, \qquad C_{\alpha\beta} = -C_{\beta\alpha} \tag{A.5}$$
$$\theta_\alpha = \theta^\beta C_{\beta\alpha}, \qquad C^{\alpha\beta}C_{\beta\gamma} = \delta^\alpha_\gamma. \tag{A.6}$$

It should be noted, however, that the conjugation matrix is symmetric in chapter 4, where we are dealing with Euclidean space.

- The totally antisymmetric tensor ϵ^{ijk} obey

$$\epsilon^{012} = 1. \tag{A.7}$$

- The Dirac gamma matrices is used extensively in this thesis and is worthy of a more lengthy explanation. When having worked a while with the matrices in various applications one is bound to notice the remarkable similarities between γ-matrices and girls, which is where the notation "γ" originates from. At first, both seem harmless and strangely attractive, in the next stage their hidden complexity is revealed and fills you with self-doubt. By patience (and trial-and-error) the final stage can be reached, in which one fully understand them and realize the many applications and plain fun involved.

Dirac matrices are generators to the Clifford algebra and satisfy the anticommutation relations,

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}. \tag{A.8}$$

Continuing the gamma/girl analogy it should be stressed that, just as two girls never commute[1], neither do different γ-matrices.

We also use a shorthand notation for the antisymmetrized product of Dirac matrices,

$$\gamma^{\mu_1\mu_2\ldots\mu_n} = \gamma^{[\mu_1}\gamma^{\mu_2}\ldots\gamma^{\mu_n]}. \tag{A.9}$$

For instance, we have,

$$\gamma^{\mu\nu} = \frac{1}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right), \tag{A.10}$$
$$\gamma^{\mu\nu\rho} = \frac{1}{6}\left(\gamma^\mu\gamma^\nu\gamma^\rho + 5\text{ terms}\right). \tag{A.11}$$

[1] Unless perhaps if they are twins.

Furthermore, Chapter 2 and 3 lives in Minkowskian spacetime with

$$\begin{array}{ll} \left(\gamma^{(0)}\right)_{\alpha\beta} & \text{antisymmetric} \\ \left(\gamma^{\mu}\right)_{\alpha\beta}, \left(\gamma^{\mu\nu}\right)_{\alpha\beta} & \text{symmetric} \\ \left(\gamma^{\mu\nu\rho}\right)_{\alpha\beta} \left(\gamma^{\mu\nu\rho\delta}\right)_{\alpha\beta} & \text{antisymmetric,} \end{array}$$

while chapter 4 inhabit Euclidean space with

$$\begin{array}{ll} \left(\gamma^{(0)}\right)_{\alpha\beta}, \left(\gamma^{\mu}\right)_{\alpha\beta} & \text{symmetric} \\ \left(\gamma^{\mu\nu}\right)_{\alpha\beta}, \left(\gamma^{\mu\nu\rho}\right)_{\alpha\beta} & \text{antisymmetric.} \end{array}$$

Also, in chapter 4 we need Dirac matrices in various dimensions, in which case [68] is a good reference and details a recursive scheme to construct Dirac matrices in arbitrary dimension.

A.2 List of quantities

Here we list quantities used throughout the thesis. In the cases where notation changes or is specific to a certain section it will be clearly noted. The quantities will be listed according to which chapter they first appeared. To avoid unnecessary repetition in the later chapters we only list quantities that have not yet appeared. Where deemed necessary a reference to a proper definition is given in parenthesis.

- **Chapter 2**

2.1 The bosonic membrane

p	:	Dimension of brane
D	:	Dimension of spacetime
$\eta_{\mu\nu}$	:	Flat Minkowski metric
ξ^i	:	Worldvolume coordinates
τ, σ^1, σ^2	:	Worldvolume coordinates; ξ^0, ξ^1, ξ^2
X^μ	:	Embedding fields
g_{ij}	:	Induced metric on the worldvolume
g	:	Determinant of g_{ij}
E_i^μ	:	Vielbein
T	:	Membrane tension
P_μ	:	Conjugate momenta

$\dot{X}^\mu, X'^\mu, \bar{X}^\mu$	:	Shorthand for various derivatives, see (2.10)
ϕ_i	:	Primary constraints for the membrane
$\{\cdot,\cdot\}$	:	Poisson brackets
$\hat{X}^\mu, \hat{P}\mu, \hat{\phi}_i$	:	Operator form of canonical variables and constraints
$X^\pm$	:	Lightcone coordinates
$\vec{X}, X^a$	:	Transverse coordinates
$\bar{g}, u_r, g_{00}$	:	Components of the induced metric in the lightcone gauge
$\vec{P}, P^+$	:	Canonical momenta conjugate to $\vec{X}$ and X^-
ϕ_r	:	Primary constraint in the Hamiltonian formalism
w_{rs}	:	Spatial metric on membrane
w	:	Determinant of w_{rs}
P_0^+	:	Membrane momentum in the X^- direction
$\vec{P}_0, P_0^-$	:	Center of mass momenta
$\mathcal{M}$	:	Membrane mass
$\vec{X}_0$	:	Zero mode
$[\,\cdot\,]'$	:	Exclusion of the zero mode

Exceptions:
2.1.5 Area preserving diffeomorphisms

$\{\cdot,\cdot\}$	:	Lie bracket (2.59)

2.2 Supersymmetry

$Q, Q^\dagger$	:	Supercharges
$\{\cdot,\cdot\}$	:	Anticommutator
R	:	Curvature (coefficient in Einstein-Hilbert term)
θ^α	:	Fermionic coordinates
Z^M	:	Superspace coordinates
N_B	:	Number of bosonic degrees of freedom
N_F	:	Number of fermionic degrees of freedom
M	:	Number of minimal spinor components
N	:	Number of supersymmetries
E_i^μ	:	Supervielbein
$\Gamma^{\mu\nu}$	:	Dirac matrices

Γ : Matrix defined in (2.89)
$\bar{\theta}$: Adjoint spinor (A.4)
$\not{E}_i$: Feynman-slashed supervielbein (A.2)
J^i : Supercurrents
Π_i^A : Pullback (2.97)
$E_M{}^A$: Supervielbein (2.98)
H : Curved superspace 4-form
B : The potential of H (2.100)

- **Chapter 3**

$\Gamma^{\pm}$: Lightcone gamma matrices
χ : Primary constraint
θ_0 : Fermionic zero mode
T : Kinetic energy
V : Potential energy
Y_A : Orthonormal set of basis functions
η_{AB} : Metric w.r.t. $Y_A{}^C$
$f_{AB}{}^C$: Structure constants
U, W : 't-Hooft clock and shift matrices
$F^{\mu\nu}$: Field strength
Ψ : 16-component Majorana-Weyl spinor
A_μ : A $U(N)$ hermitian gauge field
D_μ : Covariant derivative
g_{YM} : Yang-Mills coupling constant
Z_q : Quantum partition function
Z_{cl} : Classical partition function
ψ_t : Toy model wave function
χ : Smooth function with compact support
φ : Oscillator wave function
ξ_F : Spinor part of toy model wave function

ε	:	Arbitrarily small but positive parameter
E_p, m_p	:	Energy and mass of the Planck scale
R	:	Radius of circle closed strings wrap around
p	:	Kaluza-klein modes

- **Chapter 4**

κ	:	Exponent in wave function expansion
Ψ	:	The ground state wave function in toy model
g, f	:	Polynomial solutions to (4.17) and (4.18)
X	:	Three-dimensional Euclidean space
q	:	Bosonic coordinates (4.21)
γ^i	:	Fermionic coordinates (4.22)
s_d	:	Dimension the representation of the Clifford algebra
d	:	Dimension of the ground state model
$\mathcal{C}$	:	Representation space, see section 4.2.1
$\Theta_{\alpha A}$	:	Clifford generators (4.27)
s, t	:	indices; $1, \ldots, d$
α, β	:	indices; $1, \ldots, s_d$
$\mathcal{H}$	:	Hilbert space
$\vec{\nabla}_t$	:	Derivative w.r.t. $\vec{q}_t$
J_{AB}, L_{AB}, M_{AB}	:	Transformation generators defined in (4.32)
J_{st}, L_{st}, M_{st}	:	Transformation generators defined in (4.33)
$\vec{e}$	:	$SO(3)$ unit vector
E_s	:	$SO(d-1)$ unit vector
r	:	Introduced length variable (strictly positive)
$\vec{y}_s$	:	Transversal coordinates
ψ	:	The ground state wave function
ψ_n	:	The nth term in the asymptotic expansion of Ψ
$\lvert F(E, \vec{e})\rangle$	:	Fermionic states of the ground state wave function
$\vec{n}_\pm$	:	Complex vectors
H^0	:	Hamiltonian obtained from Q^0.
H^0_B, H^0_F	:	Bosonic and fermionic parts of H^0

u, v	:	Vectors in $\mathbb{R}^{s_d}$
φ_B^0	:	Ground state of H_B^0
P^+, P^-	:	Projection operators
U, R	:	Group elements
$\mathcal{U}, \mathcal{R}$	:	Representations of $Spin(s_d)$ and $SU(2)$, respectively

4.4 A numerical method

D	:	Dimension of the super Yang-Mills theory
$\{\cdot\}$	:	(Computer) lists
$a^\dagger, a$	:	Creation and annihilation operators
$\|n\rangle$	:	Elementary states
x	:	Bosonic coordinate operator
p	:	Momentum operator
n_s	:	Number of superposed elementary states
$\|st\rangle$	:	Arbitrary quantum state

B

Proof of κ-symmetry

For a long time the hypothesized supermembrane was in dire straits; without κ-symmetry equation (2.79) has no solution for the $p = 2$ case. It was thus of crucial importance when Hughes, Liu and Polchinski in 1986 generalized the κ-symmetry of strings to include membranes as well [3].

In this appendix we prove that the supermembrane action is invariant under κ-symmetry, by and large following chapter 4 of [69]. In contrast to the bulk of this thesis we will present the calculations made here in great detail.

B.1 Preliminaries

In this appendix we will need an easy and logical way to distinguish between curved and flat space as well as between vectors and spinors. Thus we change our previous notation in favor of

$$A, B, C, \ldots \text{ flat super indices} \begin{cases} a, b, c, \ldots & \text{flat vector} \\ \alpha, \beta, \gamma, \ldots & \text{flat spinor} \end{cases} \tag{B.1}$$

$$M, N, P, \ldots \text{ curved super indices} \begin{cases} m, n, p, \ldots & \text{curved vector} \\ \mu, \nu, \rho, \ldots & \text{curved spinor} \end{cases} \tag{B.2}$$

Hence our superspace coordinates become

$$Z^M(\xi) = (X^m, \theta^\mu)(\xi). \tag{B.3}$$

Furthermore we define the pullback

$$\Pi_i^A = (\partial_i Z^M) E_M{}^A, \tag{B.4}$$

where $E_M{}^A$ is the supervielbein. Moreover, we have the 3-form

$$B = \frac{1}{3!} E^A E^B E^C B_{CBA}, \tag{B.5}$$

with,

$$E^A = dZ^M E_M{}^A, \tag{B.6}$$

where B is the potential to the 4-form H,

$$H = dB. \tag{B.7}$$

The supermembrane action in eleven-dimensional supergravity is now

$$S = \int d^3\xi \left(-\frac{1}{2}\sqrt{-g} g^{ij} \Pi_i^a \Pi_j^b \eta_{ab} + \frac{1}{2}\sqrt{-g} - \frac{1}{3!} \epsilon^{ijk} \Pi_i^A \Pi_j^B \Pi_k^C B_{CBA} \right). \tag{B.8}$$

The κ-symmetry is then defined as

$$(\delta_\kappa Z^M) E_M^a = 0 \tag{B.9}$$
$$(\delta_\kappa Z^M) E_M^\alpha = (1+\Gamma)^\alpha{}_\beta \kappa^\beta = \tilde{\kappa}^\alpha, \tag{B.10}$$

with $\kappa^\beta(\xi)$ an anticommuting spacetime spinor and Γ, as usual,

$$\Gamma \equiv \frac{\epsilon^{ijk}}{6\sqrt{-g}} \Pi_i^a \Pi_j^b \Pi_k^c \Gamma_{abc}. \tag{B.11}$$

Furthermore we use the "1.5 formulation", with the equation of motion for g_{ij} being

$$g_{ij} = \Pi_i^a \Pi_j^b \eta_{ab}. \tag{B.12}$$

B.2 The proof

Before we prove that the above action is invariant under κ-symmetry we want to check that $(1+\Gamma)$ in B.10 is indeed a projection operator. This amounts to checking that $\Gamma^2 = 1$. We start by noting that

$$\begin{aligned} \Gamma^2 &= -\frac{1}{36g} \epsilon^{ijk} \Pi_i^a \Pi_j^b \Pi_k^c \epsilon_{lmn} \Pi_d^l \Pi_e^m \Pi_f^n \Gamma_{abc} \Gamma^{def} \\ &= -\frac{1}{36g} \epsilon^{ijk} \Pi_i^a \Pi_j^b \Pi_k^c \epsilon_{lmn} \Pi_d^l \Pi_e^m \Pi_f^n \times \\ &\quad \left(\Gamma_{abc}{}^{def} + 9\delta_{[a}{}^{[d} \Gamma_{bc]}{}^{ef]} - 18\delta_{[ab}{}^{[de} \Gamma_{c]}^{f]} - 6\delta_{[abc]}^{def} \right). \end{aligned} \tag{B.13}$$

To proceed we now rid ourselves of the first term by invoking the antisymmetry in the indices of Γ and the symmetry in the remaining expression (specifically that $\epsilon^{ijk}\epsilon_{lmn} = 6\delta^{ijk}_{[lmn]}$, and δ is symmetric). To continue the above we now have,

$$\begin{aligned} \dots &= -\frac{9}{36g}\epsilon^{ijk}\Pi_i^{[a}\Pi_j^b\Pi_k^{c]}\epsilon_{lmn}\Pi^l_{[a}\Pi^m_e\Pi^n_{f]}\Gamma_{bc}{}^{ef} \\ &\quad +\frac{18}{36g}\epsilon^{ijk}\Pi_i^{[a}\Pi_j^b\Pi_k^{c]}\epsilon_{lmn}\Pi^l_{[a}\Pi^m_b\Pi^n_{f]}\Gamma_c{}^f \\ &\quad +\frac{6}{36g}\epsilon^{ijk}\Pi_i^{[a}\Pi_j^b\Pi_k^{c]}\epsilon_{lmn}\Pi^l_{[a}\Pi^m_b\Pi^n_{c]} \\ &= \frac{6}{36g}\epsilon^{ijk}\epsilon_{ijk} = \frac{6}{36}\epsilon^{ijk}\epsilon^{ijk} = 1, \end{aligned} \tag{B.14}$$

where we in the second equality have used the same symmetry arguments as earlier and the simple fact that

$$\Pi_i^a\Pi_a^l = \delta_l^i. \tag{B.15}$$

In the third equality we have used that

$$\epsilon_{ijk} = g\epsilon^{ijk}, \tag{B.16}$$

which we easily derive from the following expression of g,

$$g = \det g_{\alpha\beta} = \frac{1}{3!}\epsilon^{ijk}\epsilon^{lmn}g_{il}g_{jm}g_{kn}. \tag{B.17}$$

We have thus convinced ourselves that $\Gamma^2 = 1$ and move on to check that the action (B.8) is κ-symmetric. This is a rather lengthy enterprise, so we choose to split our burden into two terms according to,

$$\delta_\kappa S = \int d^3\xi \left(\underbrace{-\frac{1}{2}\sqrt{-g}g^{ij}(\delta_\kappa\Pi_i^a)\Pi_j^b\eta_{ab}}_{\text{term I}} - \underbrace{\delta_\kappa(\frac{1}{3!}\epsilon^{ijk}\Pi_i^A\Pi_j^B\Pi_k^C B_{CBA})}_{\text{term II}}\right). \tag{B.18}$$

We begin by calculating

$$\begin{aligned} \delta_\kappa\Pi_i^A &= (\partial_i\delta_\kappa Z^M)E_M{}^A + \partial_i Z^M(\delta_\kappa E_M{}^A) \\ &= \partial_i\tilde\kappa^A - (\delta_\kappa Z^M)\partial_i E_M{}^A + \partial_i Z^M(\delta_\kappa E_M{}^A), \end{aligned} \tag{B.19}$$

where we immediately note that $\partial_i\tilde\kappa^A = 0$ as we are dealing with the bosonic case only ($\tilde\kappa^a = 0$). We continue the above and get

$$\begin{aligned} \dots &= -(\delta_\kappa Z^M)\partial_i E_M{}^A + \partial_i Z^M\delta_\kappa Z^N\partial_N E_M{}^A \\ &= -\tilde\kappa^M\partial_i E_M{}^A + \partial_i Z^M\tilde\kappa^N\partial_N E_M{}^A \\ &= -\tilde\kappa^M\Pi_i^N\partial_N E_M{}^A + \partial_i Z^M\tilde\kappa^N\partial_N E_M{}^A \\ &= -\tilde\kappa^M\partial_i Z^N\partial_N E_M{}^A + \partial_i Z^M\tilde\kappa^N\partial_N E_M{}^A \\ &= 2\partial_i Z^M\tilde\kappa^N\partial_{[N}E^A_{M\}} \\ &= 2\Pi_i^B E_B{}^M\tilde\kappa^N\partial_{[N}E^A_{M\}}. \end{aligned} \tag{B.20}$$

To proceed we introduce the spin connection,

$$\Omega_A{}^B = dZ^M \Omega_{MA}{}^B, \tag{B.21}$$

where $\Omega_A{}^B$ belong to the Lie algebra $\widetilde{SO(1,10)}$, which is not a superalgebra. Ergo, vectors and spinors don't mix:

$$\Omega_a{}^\beta = \Omega_\alpha{}^b = 0. \tag{B.22}$$

We also need the covariant derivative,

$$D = d + \Omega, \tag{B.23}$$

which, if acting on a tensor becomes,

$$DT^A = dT^A + T^B \wedge \Omega_B{}^A \tag{B.24}$$

$$DT_A = dT_A - \Omega_A{}^B \wedge T_B. \tag{B.25}$$

We will also need the torsion,

$$T^A = DE^A = dE^A + E^B \wedge \Omega_B{}^A. \tag{B.26}$$

We can now derive two different expressions for T^A,

$$T^A = \frac{1}{2} dZ^{M_1} dZ^{M_2} T_{M_2 M_1}{}^A \tag{B.27}$$

$$\begin{aligned} T^A &= DE^A = dZ^{M_1} \partial M_1 dZ^{M_2} E_{M_2}{}^A \\ &\quad + dZ^{M_1} E_{M_1}{}^B dZ^{M_2} \Omega_{M_2 B}{}^A, \end{aligned} \tag{B.28}$$

where the factor 1/2 is by convention. Putting these two expressions equal we obtain

$$\partial_{[M_2} E_{M_1\}}{}^A = \frac{1}{2} T_{M_2 M_1}{}^A + (-)^{BM_2} E_{[M_2}{}^B \Omega_{M_1\}B}{}^A, \tag{B.29}$$

where we have introduced the notation $(-)^E$, with the exponent $E = 0$ for a bosonic index and $E = 1$ for a fermionic index. Using (B.29) we get

$$\delta_\kappa \Pi_i^A = \underbrace{\Pi_i^B E_B{}^M \tilde{\kappa}^N T_{NM}{}^A}_{\text{term III}} + \underbrace{2(-)^{BN} \Pi_i{}^C E_C{}^M \tilde{\kappa}^N E_{[N}{}^B \Omega_{M\}B}{}^A}_{\text{term IV}}. \tag{B.30}$$

Using the relation

$$T_{NM}{}^A = (-)^{N(M+C)} E_M{}^C E_N{}^B T_{BC}{}^A \tag{B.31}$$

in term III, we get

$$\begin{aligned} III &= (-)^{N(M+C)}\Pi_i^D E_D{}^M \tilde{\kappa}^N E_M{}^C E_N{}^B T_{BC}{}^A \\ &= \Pi_i^C \tilde{\kappa}^\beta T_{\beta C}{}^A. \end{aligned} \tag{B.32}$$

To calculate IV we remind ourselves that $A = a$ and get

$$\begin{aligned} IV &= -(-)^{N(B+M)}\Pi_i^C E_C{}^M \tilde{\kappa}^N E_M{}^B \Omega_{NB}{}^a \\ &= -\Pi_i^c \tilde{\kappa}^N \Omega_{Nc}{}^a, \end{aligned} \tag{B.33}$$

which, if inserted back into I, vanish due to the antisymmetry of a and c in $\Omega_{Nc}{}^a$ and symmetry in the remaining expression. Thus we are left with

$$I = -\sqrt{-g} g^{ij} \Pi_i^C \Pi_j^a \tilde{\kappa}^\beta T_{\beta C a}. \tag{B.34}$$

This can be further simplified by using the renormalization found in [69],

$$T_{\beta C a} = 2i(\Gamma_a)_{\beta C}, \tag{B.35}$$

and the fact that $C = \gamma$ yields the only non-zero contribution (we reach this conclusion by dimensional arguments, see [69] for details). We thus arrive at

$$I = -2i\sqrt{-g} g^{ij} \Pi_i^\gamma \Pi_j^a \tilde{\kappa}^\beta (\Gamma_a)_{\beta\gamma}. \tag{B.36}$$

The next step is to calculate the term

$$\begin{aligned} \tilde{\kappa}^\beta (\Gamma_d)_{\beta\gamma} &= (1+\Gamma)^\beta{}_\alpha \kappa^\alpha (\Gamma_d)_{\beta\gamma} \\ &= \left(1 + \frac{\epsilon^{ijk}}{6\sqrt{-g}} \Pi_i^a \Pi_j^b \Pi_k^c \Gamma_{abc}\right)^\beta{}_\alpha \kappa^\alpha (\Gamma_d)_{\beta\gamma}. \end{aligned} \tag{B.37}$$

This term we divide into terms proportional to $\Gamma^{(n)}$, and we draw the conclusion that we cannot have a term proportional to $\Gamma^{(4)}$ (the worldvolume is three dimensional, hence antisymmetry in four worldvolume indices is not possible). We are left with two terms;

$$\frac{\epsilon^{ijk}}{6\sqrt{-g}} \Pi_i^a \Pi_j^b \Pi_k^c (3\eta_{ad}\Gamma_{bc}\kappa)_\gamma \tag{B.38}$$

and

$$(\kappa\Gamma_d)_\gamma. \tag{B.39}$$

This result lets us write down the final expression for term I,

$$\begin{aligned} I &= -2i\sqrt{-g} g^{lm} \Pi_l^\gamma \Pi_m^d \left(\frac{\epsilon^{ijk}}{2\sqrt{-g}} \Pi_i^a \Pi_j^b \Pi_k^c \eta_{ad}\Gamma_{bc}\kappa + \kappa\Gamma_d\right)_\gamma \\ &= -2i\sqrt{-g}\Pi_l^\gamma \left(\frac{\epsilon^{ljk}}{2\sqrt{-g}} \Pi_j^b \Pi_k^c \Gamma_{bc}\kappa + g^{lm}\Pi_m^d \kappa\Gamma_d\right)_\gamma. \end{aligned} \tag{B.40}$$

To calculate term II we need to use some topological arguments (see, e.g., [70]). We note that the κ-variation of term III (a 3-form) can be alternatively expressed as a Lie derivative, i.e.,

$$\delta_\kappa \hat{B} = \mathcal{L}_\kappa \hat{B} \equiv (di_\kappa + i_\kappa d)\hat{B} = di_\kappa \hat{B} + i_\kappa \hat{H}, \tag{B.41}$$

where i_κ is the interior derivative and the hat denotes pullback. The first term is a total derivative and the second is

$$i_\kappa \hat{H} = \frac{1}{3!} d^3\xi \, \epsilon^{ijk} \Pi_i^A \Pi_j^B \Pi_k^C \tilde{\kappa}^\gamma H_{\gamma CBA}. \tag{B.42}$$

From this result we get,

$$II = \frac{1}{3!} \epsilon^{ijk} \Pi_i^A \Pi_j^B \Pi_k^C \tilde{\kappa}^\gamma H_{\gamma CBA}. \tag{B.43}$$

From the constraints of eleven-dimensional supergravity (again, see [69]) H reduces to one term only, and we are left with

$$II = \frac{3 \cdot 2i}{3!} \epsilon^{ijk} \Pi_i^a \Pi_j^b \Pi_k^\delta \tilde{\kappa}^\gamma (\Gamma_{ba})_{\gamma\delta}. \tag{B.44}$$

Inserting the full expressions for $\tilde{\kappa}$ and Γ we end up with three terms; V, VI and VII,

$$V = i\epsilon^{ijk} \Pi_i^a \Pi_j^b \Pi_k^\delta \kappa^\gamma (\Gamma_{ba})_{\gamma\delta}, \tag{B.45}$$

Terms VI and VII both arise from keeping terms proportional to $\Gamma^{(3)}$ and $\Gamma^{(1)}$, respectively, of the remaining term,

$$\frac{i}{6\sqrt{-g}} \epsilon^{ijk} \epsilon^{lmn} \Pi_i^a \Pi_j^b \Pi_k^\delta \Pi_l^d \Pi_m^e \Pi_n^f (\Gamma_{ba} \Gamma_{def} \kappa)_\delta. \tag{B.46}$$

We note that

$$\Gamma^{ba} \Gamma_{def} = \Gamma^{ba}{}_{def} + 6\Gamma^{[b}{}_{[ef} \delta^{a]}{}_{d]} - 6\Gamma_{[f} \delta^{ba}_{de]}. \tag{B.47}$$

Consequently, term VI becomes,

$$\begin{aligned} VI &= \frac{i}{\sqrt{-g}} \epsilon^{ijk} \epsilon^{lmn} \Pi_i^a \Pi_j^b \Pi_k^\delta \Pi_l^d \Pi_m^e \Pi_n^f (\eta_{ad} \Gamma_{bef} \kappa)_\delta \\ &= \frac{i}{\sqrt{-g}} \epsilon^{ijk} \epsilon^{lmn} g_{il} \Pi_j^b \Pi_k^\delta \Pi_l^d \Pi_m^e \Pi_n^f (\eta_{ad} \Gamma_{bef} \kappa)_\delta \\ &= \frac{i}{\sqrt{-g}} (g^{jn} g^{km} - g^{jm} g^{kn}) \Pi_j^b \Pi_k^\delta \Pi_l^d \Pi_m^e \Pi_n^f (\eta_{ad} \Gamma_{bef} \kappa)_\delta \\ &= 0, \end{aligned} \tag{B.48}$$

where we in the last equality used the standard symmetry arguments. In the penultimate equality we used the relation

$$\epsilon^{ijk}\epsilon^{lmn}g_{kl} = g^{in}g^{jm} - g^{im}g^{jn}. \tag{B.49}$$

Keeping terms proportional to $\Gamma^{(1)}$ gives us,

$$\begin{aligned} VII &= -\frac{i}{\sqrt{-g}}\epsilon^{ijk}\epsilon^{lmn}\Pi_{ai}\Pi_{bj}\Pi_k^\delta\Pi_l^d\Pi_m^e\Pi_n^f(\Gamma_f\delta_{de}^{ba}\kappa)_\delta \\ &= -\frac{i}{\sqrt{-g}}\epsilon^{ijk}\epsilon^{lmn}g_{im}g_{jl}\Pi_k^\delta\Pi_n^f(\Gamma_f\kappa)_\delta \\ &= -2i\sqrt{-g}g^{kn}\Pi_k^\delta\Pi_n^f(\Gamma_f\kappa)_\delta, \end{aligned} \tag{B.50}$$

where we in the second equality used the equations of motion (B.12) and in the last equality used the relation

$$\begin{aligned} 2gg^{il} &= 2\frac{1}{3!}\epsilon^{ijk}\epsilon^{lmn}g_{il}g_{jm}g_{kn}g^{il} \\ &= \epsilon^{ijk}\epsilon^{lmn}g_{jm}g_{kn}. \end{aligned} \tag{B.51}$$

And now, finally, by collecting the relevant terms we find that $I - II = 0, i.e.,$the κ-variation of the action is zero.

C

Some explicit calculations in the $d = 3$ case

To aid our understanding of the $SU(2)$ ground state theorem we will make some explicit calculations for the case where $d = 3$ (and thus $s_d = 4$).

For the $d = 3$ case we have the Clifford generators

$$(\Theta_{\alpha A})_{\alpha=1,2,3,4;\ A=1,2,3}\,, \tag{C.1}$$

defined on $\mathcal{C} = \mathbb{C}^{4\otimes 3}$ and satisfying

$$\{\Theta_{\alpha A}, \Theta_{\beta B}\} = \delta_{\alpha\beta}\delta_{AB}. \tag{C.2}$$

To realize this algebra we call upon the Pauli sigma matrices,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{C.3}$$

These are then used to construct the Dirac gamma matrices,

$$\begin{aligned}
\gamma_1 &= \sigma^1 \otimes \mathbb{1}_2 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} \\
\gamma_2 &= i\sigma^2 \otimes \sigma^1 = \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix} \\
\gamma_3 &= i\sigma^2 \otimes i\sigma^2 = \begin{pmatrix} 0 & i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix} \\
\gamma_4 &= i\sigma^2 \otimes \sigma^3 = \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} \\
\gamma_5 &= \sigma^3 \otimes \mathbb{1}_2 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix},
\end{aligned}$$

which we note are all *real* and anticommute. These matrices now pave the way for the $\Theta_{\alpha A}$-matrices,

$$\begin{aligned}\Theta_{\alpha 1} &= \frac{1}{\sqrt{2}}\left(\gamma_\alpha \otimes \mathbb{1}_4 \otimes \gamma_5\right)\\ \Theta_{\alpha 2} &= \frac{1}{\sqrt{2}}\left(\gamma_5 \otimes \gamma_\alpha \otimes \mathbb{1}_4\right)\\ \Theta_{\alpha 3} &= \frac{1}{\sqrt{2}}\left(\mathbb{1}_4 \otimes \gamma_5 \otimes \gamma_\alpha\right),\end{aligned}$$

which are real 64×64 matrices and realize the (C.2) algebra.

To describe the fermionic states $|F(E, \vec{e})\rangle$ we form the two complex vectors $\vec{n}_\pm$, satisfying

$$\vec{n}_+ \cdot \vec{n}_- = 1 \tag{C.4}$$
$$\vec{e} \times \vec{n}_\pm = \mp i\vec{n}_\pm, \tag{C.5}$$

and hence also

$$\vec{n}_\pm \cdot \vec{n}_\pm = 0 \tag{C.6}$$
$$\vec{n}_+ \times \vec{n}_- = -i\vec{e}. \tag{C.7}$$

To write down the explicit forms of $\vec{n}_\pm$ we need an expression for the vector $\vec{e}$. To make it simple for us we set $\vec{e} = (1, 0, 0)$ (we can perform arbitrary rotations of this vector afterwards). We then make the ansatz,

$$\vec{n}_+ = (0, r(\cos\rho + i\sin\rho), r(-\sin\rho + i\cos\rho)) \tag{C.8}$$
$$\vec{n}_- = \left(0, \frac{1}{2r}(\cos\rho - i\sin\rho), -r\frac{1}{2r}(\sin\rho + i\cos\rho)\right), \tag{C.9}$$

and proceed by checking the relations (C.4)-(C.7):

$$\vec{n}_+ \cdot \vec{n}_- = \frac{1}{2}(\cos^2\rho + \sin^2\rho + \cos^2\rho + \sin^2\rho) = 1 \tag{C.10}$$
$$\vec{e} \times \vec{n}_+ = (0, r(-\sin\rho - i\cos\rho), r(\cos\rho + i\sin\rho)) = -i\vec{n}_+ \tag{C.11}$$
$$\vec{e} \times \vec{n}_- = (0, \frac{1}{2r}(\sin\rho + i\cos\rho), \frac{1}{2r}(\cos\rho - i\sin\rho)) = i\vec{n}_- \tag{C.12}$$
$$\begin{aligned}\vec{n}_+ \cdot \vec{n}_+ &= r^2(\cos^2\rho - \sin^2\rho + 2i\sin\rho\cos\rho\\ &\quad + \sin^2\rho - \cos^2\rho - 2i\sin\rho\cos\rho) = 0\end{aligned} \tag{C.13}$$
$$\begin{aligned}\vec{n}_- \cdot \vec{n}_- &= \frac{1}{4r^2}(\cos^2\rho - \sin^2\rho - 2i\sin\rho\cos\rho\\ &\quad + \sin^2\rho - \cos^2\rho + 2i\sin\rho\cos\rho) = 0\end{aligned} \tag{C.14}$$
$$\begin{aligned}\vec{n}_+ \times \vec{n}_- &= (-\frac{1}{2}(\cos\rho\sin\rho + i\cos^2 + i\sin^2\rho - \sin\rho\cos\rho\\ &\quad - \sin\rho\cos\rho + i\sin^2\rho + i\cos^2\rho + \cos\rho\sin\rho, 0, 0)\\ &= (-i, 0, 0) = -i\vec{e}.\end{aligned} \tag{C.15}$$

In the spirit of explicitness, we write down specific forms of the unit vectors $\vec{e}$ and E for the case of $d = 9$, compliments of [55],

$$\vec{e} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}, \tag{C.16}$$

and

$$E = \begin{pmatrix} \sin\epsilon_8 \sin\epsilon_7 \ldots \sin\epsilon_2 \sin\epsilon_1 \\ \sin\epsilon_8 \sin\epsilon_7 \ldots \sin\epsilon_2 \cos\epsilon_1 \\ \sin\epsilon_8 \sin\epsilon_7 \ldots \cos\epsilon_2 \\ \vdots \\ \sin\epsilon_8 \cos\epsilon_7 \\ \cos\epsilon_8 \end{pmatrix}. \tag{C.17}$$

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