Homework 13.

Problem 1.
(a) Let’s $M_M$ is mass of the Milky Way.

\[ \frac{M_\odot v^2}{R} = G \frac{M_\odot M_M}{R^2}, \quad M_M = \frac{v^2 R}{G} \approx 1.9 \times 10^{11} \text{ kg} \approx 10^{11} M_\odot \]

(b) Mass of the Earth is $M_E \approx 6 \times 10^{24} \text{ kg}$, period of the moons orbit $T \approx 28$ days, $1 \text{ AU} \approx 1.5 \times 10^{11} \text{ m}$. Again, like in (a) writing that centripetal force is a gravitational attraction we get $v^2 = G M_E / R$. Taking into account that $v = 2 \pi R / T$ we find for the distance

\[ R = \left( \frac{GM_E T^2}{(2\pi)^2} \right)^{1/3} \approx 3.9 \times 10^8 \text{ m} \approx 2.6 \times 10^{-3} \text{ AU} \]

(c) $M_\odot = 2 \times 10^{30} \text{ kg}$, mass of Jupiter $M_J = 1.9 \times 10^{27}$, period is $T \approx 11.86 \text{ years} \approx 3.7 \times 10^8 \text{ s}$, radius of the Sun $R_\odot = 7 \times 10^8 \text{ m}$. Distance from sun to center of mass is $r_s$ distance from Jupiter to c.o.m. is $r_J$

\[ M_\odot \omega^3 r_s = M_J \omega^3 r_J = G \frac{M_\odot M_J}{(r_s + r_J)^2} \]

from here we get

\[ r_s = M_J \left( \frac{GT^2}{(2\pi)^2(M_\odot + M_J)^2} \right)^{1/3} \approx 7.3 \times 10^8 \text{ m}, \quad r_J = M_\odot \left( \frac{GT^2}{(2\pi)^2(M_\odot + M_J)^2} \right)^{1/3} \approx 7.7 \times 10^{11} \text{ m} \]

Thus, the c.o.m is just outside of the Sun. The orbital speeds are: for Jupiter $v_J = 2 \pi r_J / T \approx 13 \text{ km/s}$, for the Sun $v_s = 2 \pi r_s / T \approx 12 \text{ m/s}$.

Problem 2.
(a) Gravitational potential energy is $P = -G m M_E / R$, the kinetic energy due to rotation of the Earth is $K = m \omega^2 R^2 / 2$, where $R$ is radius of the Earth and $\omega = 2 \pi / T$, $T = 24 \text{ hours}$. Thus, the total energy is

\[ E = \frac{m (2\pi R)^2}{2 T^2} - G \frac{m M_E}{R} \approx -6.2 \times 10^7 \text{ J}. \]

(b) East, along rotation of the Earth
(c) Orbiting means $mv^2 / R = G m M_E / R^2$, from here we get for kinetic energy $K = m v^2 / 2 = G m M_E / (2 R)$. The total mechanic energy is

\[ E = K - G \frac{m M_E}{R} = -G \frac{m M_E}{2 R} \approx -3.1 \times 10^7 \text{ J}. \]
(d) The radius of geostationary orbit is defined from \( m\omega^2 r = GM_E / r^2, \) \( r = \left( GM_E T^2 / (2\pi)^2 \right)^{1/3}. \)

The total energy is

\[
E = \frac{m\omega^2 r^2}{2} - G\frac{mM_E}{r} = -m \frac{1}{2^{1/3}} \left( \frac{\pi GM_E}{T} \right)^{2/3} \approx -4.7 \times 10^6 J.
\]

(e) If we would neglect rotation of the Earth, what is very reasonable see part (a), then \( mv^2 / 2 = GmM_E / R \) we get \( v = \sqrt{2GM_E / R} \approx 11.2 \text{ km/s}. \) To find correction due to rotation of the Earth we calculate equatorial speed of the rotation \( v_e = 2\pi R / T \approx 0.47 \text{ km/s}. \) Thus, if we'll launch the package to the east, along rotation of the Earth the sufficient speed would be \( v_s = v - v_e \approx 10.7 \text{ km/s}. \)

**Problem 3.**

We have a circular orbit. The centripetal force is due to "new" gravity

\[
\frac{mv^2}{r} = \frac{kM}{r^4}, \quad v = \sqrt{\frac{kM}{r^3}}
\]

correspondingly, the period of rotation is given by

\[
T = \frac{2\pi r}{v} = 2\pi \sqrt[5]{\frac{r^5}{kM}} \sim r^{5/2}.
\]

**Problem 4.**

First conceptually, energy loss leads to "fall" of a satellite to lower orbit, lower orbit means higher speed of rotation.

The total mechanical energy of an orbiting object is \( E = -GM_M / (2R), \) or in "differentials" \( \Delta E = \Delta RGM_M / (2R^2). \) The work done by a drag force is \( W = \Delta E = -F2\pi R, \) here we neglected change of the radius of the orbit. Now we can find \( \Delta R = -4\pi FR^3 / \left( GM_M \right), \) the satellite goes to the lower orbit.

Since, for orbital motion \( mv^2 / R = GmM / R^2, \) we can write \( v \sim R^{-1/2}, \) or \( \Delta v \sim -\Delta RR^{-3/2} / 2 \sim -v\Delta R / (2R), \) Here we see that for lower orbit we getting increase speed. Correspondingly for period \( T = 2\pi R / v, \) or \( \Delta T = 2\pi \Delta R / v - 2\pi \Delta vR / v^2 = 3/2 T \Delta R / R. \)

The torque is \( \tau = FR = dL / dt, \) from here we get \( \Delta L = FR\Delta T + F\Delta RT = 5/2 FT \Delta R. \)