Homework 9.

Problem 1.
(a) Conservation of momentum: \( m\vec{v}_0 = 2m\vec{v} \), thus \( v = v_0/2 \) in the same direction. Angular momentum should be also conserved: \( mv_0r = I_f\omega \). Moment of inertia of a puck regarding its center mass is \( mr^2/2 \), if we shift axes of rotation by \( r \) it will be \( mr^2/2 + mr^2 \) and since we have two pucks we get \( I_f = 3mr^2 \). So, \( \omega = v_0/(3r) \).

Initial kinetic energy is \( K_i = mv^2_0/2 \), final one is \( K_f = 2mv^2/2 + I_f\omega^2/2 = 5/6 \ K_i \). Fraction of energy lost is \( (K_i - K_f)/K_i = 1/6 \).
(b) Only \( v \) would be the same (as stuck-dead defined \( \omega = 0 \)). Difference of energy would go to stop rotation, and for energy loss we would get \( (K_i - K_f)/K_i = 1/2 \).

Problem 2.
There are no vertical motion. Horizontally the only force is force of friction \( F_f \). Equations of motion are \( Ma = F_f \) and \( I\alpha = F_fR \), where \( I = 2/5 \ MR^2 \) is moment of inertia of the ball, \( \alpha \) is angular acceleration. If the ball slides, it means that force of friction is maximal \( F_f = \mu Mg \). We get \( a = \mu g \) and \( \alpha = 5/2 \mu g/R \). No sliding condition is \( v = \omega R \) (ball i just rolling). Velocity is function of time \( v = v_i - at \) (linear decreasing), angular velocity is \( \omega = \alpha t \) (linear increasing). Thus at time \( t_r \) we should have \( v = \omega R \), or \( v_i - at_r = \alpha t_r \). Solving for \( t_r \) we get

\[
t_r = \frac{v_i}{a + \alpha R} = \frac{2 v_i}{7 \mu g}
\]

Problem 3.
Angular momentum is conserved \( I_i\omega_i = I_f\omega_f = I_i/2\omega_f \). \( \omega_f = 2\omega_i \). Kinetic energy initially is \( K_i = I_i\omega_i^2/2 \) and finally \( K_f = I_f\omega_f^2/2 = 2K_i \), it has increased. The explanation is that there was work done by the skater to change its moment of inertia (to contract hands while rotating).

If we would consider model of cylindrical skater with stick-like hands, and take 20 cm for a radius of cylinder and 1 m as a length of the hands, we would get that ratio of mass of the body to one hand is about 25. It is not terribly wrong.

Problem 4.
Initial kinetic energy is \( K_i = Mv^2_i/2 \), the final is \( K_f = Mv^2_f/2 + I\omega_f^2/2 \), where \( v_f = v_i - at_r = 5/7 \ v_i, \omega_f = \alpha t_r = 5/7 \ v_i/R \). Thus \( K_f = 5/7 \ K_i \) and energy loss (for heat) is \( \Delta K = 2/7 \ K_i \).

Work done by friction force is \( W = F_fx \), where \( x = v_it_t - at_t^2/2 = 12/49 \ v^2_i/(\mu g) \). So, for work we have \( W = 24/49 \ K_i \), it is bigger than \( \Delta K \), because part of the work goes to make the ball rotate. You can check that \( W - \Delta K = I\omega_f/2 \).
Problem 5.
Let’s split cone on disk layers of radius \( r = Ry/h \) and thickness \( dy \), where \( y \) is changing from 0 to \( h \) (we have cone with the pick at \( y = 0 \) and getting wider at the top). Moment of inertia of the disk layer is \( dI = dmr^2/2 \), where \( dm = \rho dV \) is mass of the layer, \( \rho \) is density of the cone, \( dV = \pi r^2 dy \) is volume of the layer.

Total moment of inertia is

\[
I = \int dI = \int \frac{1}{2}dmr^2 = \frac{1}{2} \int dV \rho r^2 = \frac{\pi}{2} \int_0^h dy r^2 \rho r^2 = \frac{\pi \rho R^4}{2h^4} \int_0^h dy y^2 = \frac{\pi \rho R^4 h}{10}.
\]

The volume of the cone is

\[
V = \int dV = \pi \int_0^h dy r^2 = \frac{\pi R^2 h}{3}.
\]

Thus we can write \( \rho = M/V \), or in other words

\[
I = \frac{3}{10} MR^2.
\]