We will now consider equations of hyperbolic form, in particular those that represent conservation laws, e.g.:

\[ \frac{\partial \psi}{\partial t} = - \nabla \cdot \mathbf{F} \quad \text{(continuity equation)} \]

which we will write (in one spatial dimension) as

\[ \frac{\partial \psi}{\partial t} = - \frac{\partial \mathbf{F}(\mathbf{u})}{\partial x} \]

with \( \mathbf{F} \) depending in general on \( \mathbf{u} \) and its derivative(s) of \( \mathbf{u} \), \( \mathbf{F} \) is the conserved flux. There are many equations that can be written in this form, or at least as coupled equations of this form. For example,

\[ \frac{\partial \psi}{\partial t} = - \frac{\partial \mathbf{F}(\mathbf{u})}{\partial x} \quad \mathbf{v} = \text{const.} \quad \text{(the advection equation)} \]

While solution is \( \mathbf{u} = f(x - vt) \), i.e., a wave propagating in positive \( x \)-direction for \( \mathbf{v} > 0 \) or \( \mathbf{v} < 0 \). The wave equation can be written as the system

\[ \frac{\partial \mathbf{u}}{\partial t} = - \frac{\partial \mathbf{A} \cdot \mathbf{u}}{\partial x} \quad \mathbf{F} = \mathbf{A} \cdot \mathbf{u} \]

where \( \mathbf{A} = \begin{bmatrix} 0 & -\mathbf{v} \\ -\mathbf{v} & 0 \end{bmatrix} \) and \( \mathbf{u} = \begin{bmatrix} \phi \\ \psi \end{bmatrix} \)

\[ \Rightarrow \begin{cases} \frac{\partial \phi}{\partial t} = \mathbf{v} \frac{\partial \psi}{\partial x} \\ \frac{\partial \psi}{\partial t} = \mathbf{v} \frac{\partial \phi}{\partial x} \end{cases} \]

\[ \Rightarrow \begin{cases} \frac{\partial^2 \phi}{\partial t^2} - \mathbf{v}^2 \frac{\partial^2 \psi}{\partial x^2} = 0 \\ \frac{\partial^2 \psi}{\partial t^2} - \mathbf{v}^2 \frac{\partial^2 \phi}{\partial x^2} = 0 \end{cases} \]

Another example is **Burgers' equation**, in 1 spatial dimension

\[ \frac{\partial \psi}{\partial t} + \mathbf{u} \frac{\partial \psi}{\partial x} = 0 \quad \text{or} \quad \frac{\partial \psi}{\partial t} + \frac{1}{2} \frac{\partial \psi^2}{\partial x} = 0 \quad \mathbf{F} = \frac{1}{2} \mathbf{u}^2 \]

This is a generalization of the advection equation with a speed that changes with time and space, which leads to interesting behavior, as one shall see next week! This is typical of fluid dynamics, e.g. Euler equation

\[ \frac{\partial \psi}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = 0 \quad \text{momentum conservation for a free, zero viscosity fluid} \]
Von Neumann Stability

Let's consider whether a particular scheme is stable or not, for this we resort to a very simple method of local analysis. We assume that the coefficients of the difference equations are slowly varying in space and time so they can be considered constants. In this case the solutions or eigenvalues, are all local phase-waves

\[ y_j^n = A^n e^{ikjAx} \]

where \( k \) is a wave-number and the "time-dependent" constant \( A^n \) at time \( n \) equals \( e^{ikjAx} \). Advancing the solution one time step

\[ y_j^{n+1} = A_{n+1} e^{ikj(Ax + Ax)} = A_n e^{ikjAx} \]

where \( \alpha = \frac{A_{n+1}}{A_n} \) is the amplification factor, which tells us how the time dependence of a simple eigenvalue is given by successive powers of the complex number \( \alpha(k) \). Therefore, the difference equations are unstable if for some \( k < |\alpha| < 1 \), this is the von Neumann stability criterion.

Let's take the advection equation and apply FTCS scheme:

\[ \frac{y_j^{n+1} - y_j^n}{\Delta t} = -\nu \frac{y_{j+1}^n - y_{j-1}^n}{2Ax} \]

Let \( \mu = \frac{\nu \Delta t}{Ax} \), then \( y_j^{n+1} = y_j^n - \mu \left[ y_{j+1} - y_{j-1} \right] \)

\[ a = \alpha \, e^{ikjAx} = \mu \left[ e^{ikjAx} - e^{-ikjAx} \right] \]

\[ \mu = 1 - i \mu \sin(kAx) \]

But \( |\alpha| > 1 \) for all \( k \)'s, so this scheme is unconditionally stable. Thus we see here that FTCS does not work at all, no choice of \( \mu \) can make it stable

Lax Method: There is a simple change that can make previous scheme stable without change:
\[ y_j^m \rightarrow \frac{1}{2} (y_{j+1}^m + y_{j-1}^m) \quad \text{(average over neighbors)} \]

\[ y_j^m = \frac{1}{2} (y_{j+1}^m + y_{j-1}^m) - \frac{\nu}{2} (y_{j+1}^m - y_{j-1}^m) \]

\[ a_{\text{Markov}} = A e^{i k_0 x} \left( e^{-i k_0 x} + e^{i k_0 x} \right) - i \mu \frac{\partial}{\partial x} A e^{i k_0 x} \sin(k_0 x) \]

\[ a = \cos(k_0 x) - i \mu \sin(k_0 x) \]

\[ \Rightarrow \text{stable for } \mu \leq 1 \quad \text{(again, Courant condition)} \]

Notice the Courant number \( \mu \) is the ratio of spatial propagation in time \( \Delta t \) to the resolution \( \Delta x \). What happens typically when \( \mu > 1 \) is that we take a \( \Delta t \) so large that for given \( v \) this requires information from gridpoints away \( \Delta t \) from where we are and, if \( \Delta x \) is smaller than this (and our scheme only uses nearest neighbors in each direction of course) we have instability. \[ \text{Notice this is a simplified picture, for complicated difference schemes things can be more complicated.} \]

To understand how a simple replacement leads to abrupt change in instability, let's rewrite the discretization in the form:

\[ y_j^{m+1} - y_j^m = \frac{1}{2} (y_{j+1}^m - y_{j-1}^m) - \frac{\nu}{2} (y_{j+1}^m - y_{j-1}^m) \]

\[ \frac{y_j^{m+1} - y_j^m}{\Delta t} = -v \frac{y_{j+1}^m - y_{j-1}^m}{2 \Delta x} + \frac{1}{\Delta t} \frac{y_{j+1}^m - y_{j-1}^m}{\Delta x^2} \]

which is FTCS discretization of

\[ \frac{\partial y}{\partial t} = -v \frac{\partial y}{\partial x} + \frac{\Delta x^2}{\Delta t} \frac{\partial^2 y}{\partial x^2} \]

Thus, we can see that the Lax scheme adds a diffusion term or, in the language of Navier Stokes equations, the Lax scheme has numerical viscosity. We can see this also from the amplification factor \( \exp(\sqrt{\nu} \Delta t) \) and the amplification of the waves decrease over time.
Note though that for $k\Delta x < 1$, the region that makes sense to expect that we can model accurately (large distances compared to grid spacing), $|k\Delta x| \ll 1$ (both in lax and FTCS schemes). However, in the FTCS case the high $k$ mode, with $k\Delta x \approx 1$ will blow up and contaminate the large-scale solution we are interested in. By adding numerical viscosity, the lax scheme actually suppresses the high-$k$ mode, so they die out and don't contaminate the desired solution at large scale, we are interested in (of course one does not trust the lax scheme when $k\Delta x$ approaches unity).

Another issue (less serious than instability, but once scheme is stable, it should be, it becomes important) is phase of $a$, not just amplitude. Phase errors lead to dispersion, for example we can rewrite "lax $a" as

$$a = e^{i\Delta x \xi} + i(1-\mu) \sin k\Delta x$$

Take an initial condition as a wave packet, which is a superposition of many $k$-modes. If $\mu = 1$, then the discretization produces the exact solution $f(x-\xi t)$ [since, remember, shift by $\Delta x$ produces phase shift $e^{-i\Delta x \xi}$ for a $k$-mode, if $\mu = 1$, this is $e^{-i\Delta x \xi}$]. However, if $\mu < 1$, there are phase errors, and the wave packet disperses as time goes on. Again, note that $(1-\mu)$ term is multiplied by $\sin k\Delta x$, so dispersion is severe once $k\Delta x$ is of order unity.

For wave equations propagation errors (in amplitude or phase) are usually not important, however, for advection equations, transport errors are more worried, in this following sense. If $v > 0$, only grid point $j$ should affect $j+1$ a time-step later, but in lax scheme something at $j$ affects both $j+1$ and $j-1$. 

To fix this one can use upwind differencing, i.e. take spatial derivatives depending on the sign of the velocity,

\[
\frac{u_j^{n+1} - u_j^n}{\Delta t} = -v_j^n \left\{ \begin{array}{c}
\frac{u_j^{n+1} - u_{j-1}^n}{\Delta x} \quad , v_j^{n+1} > 0 \\
\frac{u_j^{n+1} - u_{j}^n}{\Delta x} \quad , v_j^{n+1} < 0
\end{array} \right.
\]

Now this is only first order in space rather than second, but it does respect the symmetries of the solution we expect. The amplification factor (assuming constant \( u \)) is

\[
|A| = 1 - \frac{v}{\Delta t} (1-\mu) (1-\cos k \Delta x)
\]

which leads to the Courant condition \( \mu \leq 1 \).

**Second-order in Time Scheme**

For stable scheme, the Courant condition is not the limiting factor that other computational cost. To achieve good time resolution in a first-order in time scheme but second order in space, one has to take typically small time steps compared to \( \Delta x/v \) to achieve good accuracy. However, there are schemes that are second order in both time and space, and they can be pushed right to their stability limit, with better computational cost.

The staggered leapfrog, is one such scheme in which one computes the fluxes \( F_j^n \) from \( u_j^n \), and then computes the time derivative using time-centered differences at \( u_j^n \):

\[
\frac{u_j^{n+1} - u_j^n}{2\Delta t} = -\frac{(F_{j+1}^n - F_j^n)}{2\Delta x}
\]

for \( F = v \bar{u} \), we have

\[
u_j^{n+1} - u_j^{n-1} = -\mu \left( u_j^{n+1} - u_j^{n-1} \right) \quad \mu = \frac{v \Delta t}{\Delta x}
\]

Note that time levels on LHS shift over time \( n \) (this leapfrog) involved in RHS. This method requires two time steps to proceed in next.
The von Neumann analysis of stability now leads to a quadratic equation for $a$,

$$a - \frac{1}{a} = -2i\mu \sin k\Delta x$$

$$\Rightarrow a = -i\mu \sin k\Delta x \pm \sqrt{1 - \mu^2 \sin^2 k\Delta x}$$

and again $|a| \leq 1$ requires $|\mu| \leq 1$. Note however that as long as $|\mu| \leq 1$, $|a|^2 = 1 - \mu^2 = \frac{1}{4} |1 - e^{i\pi k\Delta x}|^2$.

Thus staggered leapfrog does not lead to amplitude dissipation.

For equations more complicated than the one we deal with here, leapfrog can lead to instabilities when there are large gradients. The Lax-Wendroff scheme is a second-order in time method that behaves better at the cost of a little bit of dissipation (but smaller than Lax).

Here, we define intermediate values at half time steps to the half grid points $x_{j+1/2}$, obeying the lax scheme

$$u_{j+1/2}^n = \frac{1}{2} (u_{j+1}^n + u_j^n) - \frac{\mu}{2} (F_{j+1}^n - F_j^n)$$

Using these values, one can compute fluxes $F_{j+1/2}^{n+1/2}$ (remember $F$ is related to $u$).

Then, the updated value of $u_j^{n+1}$ are calculated by the centered difference:

$$u_j^{n+1} = u_j^n - \mu \left( F_{j+1/2}^{n+1/2} - F_{j-1/2}^{n+1/2} \right)$$

Notice that we discard then the $u_{j+1/2}^{n+1/2}$ are discarded. For $F = \nabla u$, we have

$$u_j^{n+1} = u_j^n - \mu \left[ \frac{1}{2} (u_{j+1}^n + u_j^n) - \frac{\mu}{2} (u_{j+1}^n - u_j^n) - \frac{1}{2} (u_j^n - u_{j-1}^n) + \frac{\mu}{2} (u_j^n - u_{j-1}^n) \right]$$

which leads to:

$$a_j = 1 - i\mu \sin k\Delta x = \mu^2 (1 - \cos k\Delta x)$$

$$\Rightarrow |a_j|^2 = 1 - \mu^2 (1 - \mu^2) (1 - \cos k\Delta x)^2 \Rightarrow \mu \leq 1 \text{ for } (a_j)^2$$

However, the amplitude damping for small $k\Delta x$ is:

$$|a_j|^2 \approx 1 - \mu^2 (1 - \mu^2) \frac{(k\Delta x)^2}{4} \text{ [instead of Lax method, for which is } O((k\Delta x)^2)}$$