Hyperbolic and Dispersive Waves

There is no precise definition of wave, although one can intuitively characterize a wave as any recognizable signal that is transferred from one place to another with some characteristic velocity of propagation. The signal may disturb, change its magnitude, and velocity, provided it is still recognizable.

One can distinguish two main classes of waves, the first is formulated mathematically in terms of hyperbolic PDEs (Hyperbolic waves), the second is harder to characterize, but simple cases correspond to linear dispersive waves and we shall refer to them both as Dispersive waves. These classes are not exclusive, there is overlap in that some cases will exhibit both types of behavior, and there are some exceptions that fit neither of them-

The prototype for hyperbolic waves is the wave equation and the advection equation, which we discussed last week.

Dispersive waves are better characterized by solutions rather than type of equation. For example, a linear dispersive system admits solutions of the form,

$$q = A \cos(kx - wt)$$

where \( w \) is a real function of \( k \), the function \( w(k) \) being known as “dispersion relation.” The system is dispersive as long as \( w''(k) \neq 0 \), that is, the dispersion relation is non-linear. In this case the phase speed \( \frac{\omega(k)}{k} \) depends on \( k \), and different Fourier modes propagate at different speeds, thus they will disperse.

In majority of wave motions seem to fall into dispersive waves, the most familiar of which, ocean waves, we shall discuss next class.

In this case, the physics is described by the Laplace equation with some strange boundary conditions at the free surface.
we go back again at the conservation equation,

$$\frac{\partial q}{\partial t} + \frac{\partial F}{\partial x} = 0$$

were $F$ is the flux of $q$, $F_{(x)}$ - If $F_{(x)} = c_0 f(x)$ with $c_0$ a constant,
this becomes the advection equation with solution $y = f(x - c_0 t)$ where
the arbitrary function $f(x)$ is obtained by matching to IC or BC. If $t$ is arbitrary, we can still write the equation as

$$\frac{\partial y}{\partial t} + c(y) \frac{\partial y}{\partial x} = 0$$

were $c(y)$ is some function of $y$. One approach to solving this is to
note that it can be written as the total derivative of $f$ along a
curve $c$ that has a slope in $(x, t)$ plane given by

$$\frac{dx}{dt} = c(y) \quad (*$$

Along any curve in the $(x, t)$ plane, we can consider $x$ and $y$ to be
functions of $t$ and the total derivative of $y$ is given by

$$\frac{df}{dt} = \frac{dy}{dt} = \frac{\partial y}{\partial t} + \frac{\partial y}{\partial x} \frac{dx}{dt}$$

if the curve satisfies $(*)$, we have then

$$\frac{dp}{dt} = 0$$

along $c$, thus $f$ is a constant along $c$. Therefore, from $(*)$ we
see that the curve $c$ are straight lines in the $(x, t)$ plane with
slope $c(y)$ - Thus, the general solution of the PDE depends
on the construction of a family of straight lines in the $(x, t)$ plane,
each of them with slope $c(y)$ corresponding to the value of $f$ on the
curve. For example, let's take the IC,

$$f(x, 0) = f(x) \quad \text{for } x < 0$$
Given a point \( x = \xi \) at \( t = 0 \), one can calculate \( f(\xi) \) \( \tag{3} \)

and from the PDE the function \( f' \) is known, so one gets \( c[f(\xi)] \), let

\[
F(\xi) = c[f(\xi)] \quad \text{(slope)}
\]

then the equation for the characteristic \( \xi \) that starts from \( x = \xi \) at \( t = 0 \) is simply,

\[
\begin{align*}
\frac{d\xi}{dt} &= f'(\xi) \\
\frac{dx}{dt} &= f'(\xi) \frac{d\xi}{dx}
\end{align*}
\]

This is the solution of the problem, indeed, \( p \) can be thought as given by \( f(\xi) \) with \( \xi = \xi(t, x) \) defined implicitly by the first equation.

That is,

\[
\begin{align*}
\frac{d\xi}{dt} &= f'(\xi) \\
\frac{dx}{dt} &= f'(\xi) \frac{d\xi}{dx}
\end{align*}
\]

Now: \( \frac{d}{dt} \left[ x = \xi + F(\xi) t \right] \) leads to

\[
0 = \frac{dx}{dt} (1 + F'(t)) + F \
\Rightarrow \frac{d\xi}{dt} = - \frac{F(\xi)}{1+F'(t)}
\]

and doing \( \frac{d}{dx} \), get

\[
1 = \frac{d\xi}{dx} (1 + F'(t)) \
\Rightarrow \frac{d\xi}{dx} = \frac{1}{1+F'(t)}
\]

\[
\Rightarrow \begin{cases}
\frac{dp}{dt} = - \frac{F(\xi) f'(\xi)}{1+F(\xi) t} \\
\xi(t) \frac{dx}{dt} = F(\xi) \frac{d\xi}{dx} = \frac{F(\xi) f'(\xi)}{1+F(\xi) t}
\end{cases}
\]

\[
\Rightarrow \frac{dp}{dt} + c(p) \frac{dp}{dx} = 0 \quad /\!
\]

and the IC is satisfied, because \( x = \xi \) when \( t = 0 \), then \( p = f(\xi) \) at \( t = 0 \).

We can see from this solution that we may have problems, if \( f'(t) > 0 \), since \( 1 + F(\xi) t \) can vanish for some time \( t > t_0 \).

This corresponds to the breaking time, when solution become multi-valued.
Let's look at this. Suppose we have the following characteristics.

\[ t = \gamma x \]

where \( \gamma \) is positive for a while (in \( t \)) then it becomes negative (later). The corresponding diagram for the solution \( f(x) \) at different times would be something like this,

\[ (we \ assumed \ c'(\gamma) > 0) \]

The solution can be constructed at each time by moving each point on the initial condition \( f = f(x) \) a distance \( c(\gamma) t \) to the right, the distance moved being different for each value of \( \gamma \). Since we assumed \( c'(\gamma) > 0 \), higher values of \( \gamma \) propagate faster, thus producing a nonlinear distortion of the wave as it propagates.

When \( c'(\gamma) < 0 \), the distortion is opposite, lower values of \( \gamma \) propagate faster, probably when \( \gamma \) is constant (linear advection). All values of \( \gamma \) propagate at same speed and the shape of the wave is preserved.

Now, we see that at \( t = T \), the profile develops an infinite slope, after that the solution "breaks" and becomes a step-like-valued solution. Breaking first occurs on the characteristic \( x = \gamma t \).
for which \( F'(x) < 0 \) and \( |F'(x)| \) is a maximum, give

\[
t = \frac{-1}{F'(x)}
\]

let's take some examples. Suppose the initial distribution has a discontinuous step with \( (\phi) \) being larger behind the discontinuity than ahead,

\[
f(x) = \begin{cases} 
  \phi_1 & x \geq 0 \\ 
  \phi_2 & x < 0 
\end{cases} \quad (\phi_1 > \phi_2)
\]

with \( \phi_2 > \phi_1 \). In this case breaking happens immediately:

\[
\begin{array}{c}
\text{characteristics} \\
\text{boundary of multi-valued region}
\end{array}
\]

The boundary of the multi-valued region is given by \( x = c_2 t \) and \( x = c_1 t \), i.e. \( c_1 t < x < c_2 t \). On the other hand if we reverse the conditions \( c_2 < \phi_1 \) (and \( c_2 \phi_2 \)), though this is not necessary, there is a perfectly good continuous solution, in which all values of \( F \) are taken on characteristics that start from the origin at \( t = 0 \):
In other words, the function $F(x)$ is a step function but we use all values between $c_2$ and $c_1$ or "the face of the step" and take them all to correspond to $\xi = 0$, i.e.

$$F = x = F + c_2 \xi \leq c_1$$

$$\Rightarrow c = \frac{x}{\xi} \text{ for } \xi \leq x \leq c_1$$

In other words, the speed is

$$c = \begin{cases} \frac{x}{\xi} & \xi \leq x \leq c_1 \\ c_2 & x < \xi \leq c_1 \\ c_1 & \frac{x}{\xi} > c_1 \end{cases}$$

Thus, things look like

For the previous case, we can do the same thing, and yet $(c_2 > c_1)$

Now, in most physics situations when this type of problem arises, $\rho$ is some density and it is intrinsically single-valued. Therefore, when breaking occurs the PDE we are using is no longer valid. However, it turns out that by allowing discontinuous into the solution one can recover a single-valued solution, this replaces the multi-valued solution.
Let's go back to basic conservation law if \( \rho(x) \) is the density and \( q(x,t) \) is the flux \[ \text{assumptions}\] assuming the mass is conserved, the rate of change of mass must be due to net flow:
\[
\frac{d}{dt} \int_{x_2}^{x_1} \rho(x,t) \, dx = q(x_1,t) - q(x_2,t)
\]

As \( x_2 \to x_1 \)
\[
\frac{df}{dt} + \frac{d}{dx} q = 0
\]

This is convention for shocks.

When it is reasonable to have a relationship between \( q \) and \( \rho \),
\[
q = G(\rho)
\]
then we recover
\[
\frac{df}{dt} + G(\rho) \frac{d}{dx} \rho = 0
\]

with \( C(\rho) = \frac{dG}{d\rho} \)

Now, breaking requires us to reconsider the assumption that \( f \) and \( q \) have derivatives and that \( q = G(\rho) \) is a good approximation. Provided the continuous approximation is OK, however, we insist on the validity of the conservation law in integral form (\(*\)).

Let's allow for a simple jump discontinuity in \( q \), and see what restrictions arise from (\(*\)). Assume there is a discontinuity at time-dependent position \( x = s(t) \), and take \( x_1 \) and \( x_2 \) to be such \( x_2 < s(t) < x_1 \).

Assuming \( q \) and \( \rho \) are continuous, the first derivative, we continue for \( x_2 < x < x_1 \), and have finite limits as \( x \to s(t) \) from above and below, we have:

\[
q(x_1,t) - q(x_2,t) = \frac{d}{dt} \int_{x_2}^{s(t)} p(x,t) \, dx + \frac{d}{dt} \int_{s(t)}^{x_1} p(x,t) \, dx
\]

\[
= \int_{s(t)}^{x_1} \frac{\partial}{\partial t} p(x,t) \, dx + \int_{x_2}^{s(t)} \frac{\partial}{\partial x} q - \frac{\partial}{\partial x} p \, dx + \int_{s(t)}^{x_1} \frac{\partial}{\partial x} q - \frac{\partial}{\partial x} p \, dx
\]
Since $\frac{df}{dt}$ is bounded in each of the intervals, the integrals go to zero as $t_2 \to s^-$ and $t_1 \to s^+$, then

$$\int_{t_2}^{t_1} \left( f(s^- t) - f(s^+ t) \right) dt = \int_{t_2}^{t_1} \left( f(s^- t) - f(s^+ t) \right) dt$$

Using conventional notation that $f^-$ is ahead of shock and $f^+$ is behind, and denoting by $\frac{\partial p}{\partial t}$ the shock velocity $s$, we have

$$\frac{q_2 - q_1}{t_2 - t_1} = \frac{\partial p}{\partial t} = \frac{\partial p}{\partial t}$$

Thus, we can extend the solutions of the integral conservation law by allowing discontinuities that satisfy this condition. In the continuous part of the solution, the assumption $f^+ = f^-$ may be used, giving $q_2 = q(t_2)$ and $q_1 = q(t_1)$ on the two sides of any shock, and this gives the shock velocity through

$$U = \frac{q(t_2) - q(t_1)}{t_2 - t_1}$$

The problem then reduces to fitting shock discontinuities into the solution given by characteristics above $v(t_2 > c_1)$ in such a way that this condition is satisfied and multivalued solutions are avoided. In this case $c_2 > c_1$, for

$$\begin{cases} f = f_1 & \text{if } t \to -\infty \text{ and } x > 0, \quad t = 0 \\ f = f_2 & \text{if } t \to 0 \text{ and } x < 0 \end{cases}$$

The solution will be $f = f_1$ when ahead of shock $x > Ut$ and $f = f_2$ for $x < U t$.

Now, let's assume a quadratic relationship,

$$Q(p) = x(p^2 + \beta p + \gamma) \Rightarrow C(p) = \frac{d\tilde{Q}}{dp} = 2\alpha p + \beta$$

Thus,

$$\tilde{Y} = a(p_1 + p_2) + \beta = \frac{C_1 + C_2}{2}$$