1) Consider the 2D Ising model on a square lattice with nearest- and next-nearest-neighbor interactions, with couplings \( K \) and \( L \) respectively (see Fig. 1).

a) Carry out a decimation of this system by tracing over the white sites, leaving the black sites (i.e. lattice is renormalized by \( b = \sqrt{2} \)).

b) Calculate the interactions on the decimated lattice, keeping only terms up to \( O(K^2) \) and \( O(L) \), showing that to this order there are only two such interactions with renormalization,

\[
K' = 2K^2 + L, \quad L' = K^2. \tag{1}
\]

c) Find the fixed points for this renormalization transformation. Identify the critical fixed point and show that the critical exponent \( \nu = \ln \sqrt{2} / \ln[(2 + \sqrt{10})/3] \).

![Figure 1: 2D Ising model with nearest (K) and next-nearest (L) neighbor interactions.](image)

2) Here we explore the model in Problem 1 using numerical simulations.

a) Write a Metropolis code to perform a simulation of the 2D Ising model. Assume periodic boundary conditions. You should use lattice sizes large enough to avoid finite volume effects and beware of correlated samples (make sure your results are reasonably insensitive to your final choice).

b) Compute the power spectrum from your simulations with \( L = 0 \) at criticality and measure the critical exponent \( \eta \). Compare with Mean Field Theory and Onsager’s exact solution.
c) Write a code based on the Wolff algorithm. Check your answers for previous part agree with those obtained with the Metropolis algorithm.

d) Use your new code to explore the $L \neq 0$ model, assuming $K = 0$. Look for criticality using the order parameter and the results of Problem 1 as a starting guess. Check universality by comparing $\eta$ in this model to your result in b).

e) Use your code in c) to perform a Monte Carlo Renormalization Group simulation to measure the critical exponent $\nu$. How does this compare with the analytic result in Problem 1?