Recall special relativity (for those who didn’t have GR),
\[ ds^2 = c^2 dt^2 - dx_1^2 - dx_2^2 - dx_3^2 \quad \eta_{\mu\nu} = (1,-1,-1) \]
In the case of homogeneous and isotropic universes, the metric is instead:
\[ ds^2 = c^2 dt^2 - a(t)^2 \left( \frac{dr^2}{1-r^2} + r^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \right) = g_{\mu\nu} dx^\mu dx^\nu \]
Is proper time measured by comoving observer, with \( r, \theta, \phi = \text{const.} \)

The angular part is actually written as:
\[ r^2 \, dr^2 + r^2 \sin^2 \theta \, d\phi^2 \]

\[ g_{\mu\nu} = \left( \begin{array}{ccc}
    c^2 & -a^2 & -a^2 r^2 \\
    -a^2 & 1/r^2 & -a^2 r^2 \\
    -a^2 r^2 & -a^2 r^2 & r^2 \sin^2 \theta
\end{array} \right) \]

Sometimes (or most times), we shall set \( c = 1 \). Also, it is convenient to define conformal time \( \tau \) so that
\[ a(t) c^2 d\tau^2 = c^2 dt^2 \quad \Rightarrow \quad d\tau = \frac{dt}{a} \]
So that the metric becomes:
\[ g_{\mu\nu} = a(t)^2 \tilde{g}_{\mu\nu} \quad \text{where } \tilde{g}_{\mu\nu} \text{ is independent of } \tau \]
This defines a conformal frame formation (preserves angles)
Alternative ways of writing the spatial part of metric is:

\[ \frac{\partial^2}{\partial t^2} \left[ -c^2 \left( \frac{\sin \chi}{\sinh \chi} \right)^2 \left( \frac{d\theta^2 + \sin^2 \theta \, d\phi^2}{\chi^2} \right) \right] = \frac{2}{1-kr^2} \]

Kinematics (understand effects of expansion without solving Einstein Eqs.)

Light propagation: \( ds^2 = 0 \)
take a radial trajectory \( ds = dr = 0 \)

\[ dt^2 = \frac{A^2(t)}{1-kr^2} \, dr^2 \]

emission @ \( t = t_0, \, r = r_0 \), arrives @ \( r = 0 \) \( t = t_0 \)

\[ \int_{t_0}^{t} \frac{dt}{a(t)} = \int_{r_0}^{r} \frac{dr}{\sqrt{1-kr^2}} = \Phi(r) \]

emission @ \( t = t_0 + \delta t \), arrives @ \( t = t_0 + \delta t \)

\[ \int_{t_0}^{t} \frac{dt}{a(t)} = \Phi(r) \Rightarrow \int_{0}^{t_0 + \delta t} \frac{dt}{a_0} = \int_{0}^{t_0 + \delta t} \frac{dt}{a(t)} \]

small \( \delta t \)

\[ \frac{\delta t_0}{a_0} = \frac{\delta t}{a_1} \Rightarrow \frac{\lambda_1}{\lambda_0} = \frac{a_1}{a_0} \]

So, there are 2 affects, wavelength of light is stretched by the expansion of the universe, \( \Lambda > \lambda_0 \) if \( a_0 > a_1 \); also there is a time delay in the arrival of photons \( \delta t_0 > \delta t \).

Redshift:

\[ 1 + z = \frac{\Lambda_0}{\lambda_1} = \frac{a_0}{a_1} \]

\( z \geq 0 \) in expansion

\( z \leq 0 \) for contraction

Value shift
So, each photon of energy \( h \nu \rightarrow h \nu \frac{a_1}{a_0} \) arrives at intervals \( \Delta t_0 = \frac{\Delta t_1}{a_0} \) at detection time, the fraction of area covered by detector \( \Delta A \) is

\[
\frac{\Delta A}{4\pi a_0^2 r_1^2}.
\]

Then power at detector is

\[
P = L \left( \frac{a_1}{a_0} \right)^2 \frac{\Delta A}{4\pi a_0^2 r_1^2}.
\]

Apparent flux

\[
\Phi = \frac{P}{A} = \frac{L}{4\pi a_0^2 r_1^2 (1+z)^2}.
\]

Luminosity distance:

\[
4\pi a_L^2 \Phi = L \Rightarrow d_L = r_1 a_0 (1+z)
\]

We can write an expression for \( d_L \) as a function of \( z \) as follows:

Recall \( f(t_1) = \int_0^{t_1} \frac{dt}{\sqrt{1-hr^2}} = \left\{ \begin{array}{ll}
\sin^{-1} r_1 & = r_1 + r_1^3/6 & h=1
\end{array} \right. \]

\[
\sinh^{-1} r_1 = r_1 - r_1^3/6 & h=-1
\]

Also \( f(t, t_0) = \int_{t_0}^t \frac{dt}{a(t)} \)

\[
a(t) = a_0 + \dot{a}_0 (t-t_0) + \frac{\ddot{a}_0}{2} (t-t_0)^2 + \cdots
\]

\[
\Rightarrow \frac{a(t)}{a_0} = 1 + H_0 (t-t_0) - \frac{1}{2} q_0 H_0^2 (t-t_0)^2 + \cdots
\]

\[
H_0 = \frac{\dot{a}_0}{a_0} \text{ Hubble's constant}
\]

\[
q_0 = -\frac{\ddot{a}_0}{a_0^2 H_0^2} \text{ and } q_0 \text{ : deceleration parameter}
\]

\[
\Rightarrow H_0 d_L = \frac{\dot{a}_0}{2} \left( 1 - q_0 \right)^{3/2} + \cdots
\]

\[
H_0 \sim (z) \ll 1
\]
- So, if we know the absolute luminosity (from standard candles) and measure $I$, we get $dL$ as a function of $I$, then can get $q_0$, which tells us whether the universe is accelerating or not (more on this later when discussing SNIa)

**Dynamics**

The dynamics of the expansion is a solution to Einstein's eqs,

$$\begin{align*}
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R &= 8\pi G T_{\mu\nu} \\
\text{Peeve tensor} &\quad \text{Peeve scalar} \\
\text{strcut-energy} &\quad \text{Cosmological constant}
\end{align*}$$

Additional equation is conservation of energy-momentum $\nabla^\mu T_{\mu\nu} = 0$

For FRW metric, consistency requires $T_{\mu\nu}$ to be diagonal, and its isotropic spatial parts must be the same, e.g. perfect fluid

\[ T_{\mu\nu} = \begin{pmatrix} f(t) & \rho \phi(t) \\
-\rho \phi(t) & -p(t) \phi(t) \\
-\rho \phi(t) & -p(t) \phi(t) \end{pmatrix} \]

\[ \nabla_\nu (T_{\mu\nu}) = 0 \Rightarrow \frac{d}{dt} (\rho a^3) = -p \frac{d}{dt} (a^3) \]

\[ \text{change in energy of \ nul \ element} \]

Actually,

\[ T_{\mu\nu} = \rho g_{\mu\nu} + (\delta_{\mu\nu} + \gamma_{\mu\nu}) \gamma_{\nu\mu} \]

\[ \begin{cases} 
\gamma_{00} = 1 \\
\gamma_{ij} = 0 
\end{cases} \]

\[ \gamma = \frac{\dot{a}}{a} \]

\[ \Delta \text{ is an \ fluid} \]

\[ \text{with } \rho = \frac{\Delta}{8\pi G} \]

This simple conservation law leads to time-dependence of $\rho$ in a

dependent on Equation of state:
For $p = \omega \rho$ 

$\Rightarrow \rho \propto a^{-3(1+\omega)}$

- Matter: $p > 0$ (recall $p \propto a^{-3}$)
  $\Rightarrow \omega > 0 \Rightarrow \rho_m \propto a^{-3}$

- Radiation: $p = \frac{1}{3} \rho$ 
  $\omega = \frac{1}{3} \Rightarrow \rho_r \propto a^{-4}$

- Vacuum energy: $p = -\rho$ 
  $\Rightarrow \omega = -1 \Rightarrow p = \text{const.}$

Only two of Einstein's eqs are relevant [00 and ii], by isotropy and charge symmetry.

00: $\frac{\dot{a}^2}{a^2} + \frac{1}{a^2} = \frac{8\pi G}{3} \rho + \frac{\Lambda}{3}$

Friedmann Eqn.

"Cons. of energy" $\Rightarrow$ closed, open, flat

ii: $2 \frac{\dot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = -8\pi G \rho + \Lambda$

These two eqs + $\nabla^2 \Phi = 0$, only 2 are independent due to Bianchi's identity.

ii-00: $2 \frac{\dot{a}}{a} = -8\pi G (\rho + 3p/2) \Rightarrow \frac{\dot{a}^2}{a^2} = -\frac{8\pi G}{3} (8+3p) + \frac{\Lambda}{3}$

$\Rightarrow (8+3p)$ determines whether the universe is accelerating or decelerating ($\dot{a} > 0$)

Since $\dot{a} > 0$ today, if $8+3p > 0$ always $\Rightarrow a \rightarrow 0$ 

"Big Bang"

Matter: $p_m \propto a^{-3}$ 
$\Rightarrow a \propto t^{2/3}$ 
$\Rightarrow e^{\frac{2}{3}(1+\omega)t}$

Radiation: $p_r \propto a^{-4}$ 
$\Rightarrow a \propto t^{1/2} \times e^{\frac{1}{2}(1-\omega)t} \Rightarrow e^{\frac{1}{2}t \omega (w=-1)}$ 

[In general, these are approximate solutions]
the scale $H = \frac{a}{a}$

\[ \Rightarrow \frac{H}{a} = \frac{8\pi G}{3} \rho - \frac{\dot{a}}{a} \]

\[ \frac{\dot{a}}{H^2 a^2} = \frac{8\pi G}{3H^2} \rho - 1 \equiv \frac{1}{\text{const.}} - 1 \]

$\text{Planck} = \frac{3H^2}{8\pi G} = 1.879 \ h^2 \times 10^{-29} \frac{g}{\text{cm}^3}$ (today!)

Sign of $k$ is determined by $\frac{S}{\text{Planck}} \equiv \Omega_{\text{tot}}$

\[
\begin{align*}
\Omega_{\text{tot}} = 4 & \Rightarrow k = 0 \quad \text{(flat)} \\
\Omega_{\text{tot}} < 1 & \Rightarrow k > 0 \quad \text{(open)} \\
\Omega_{\text{tot}} > 1 & \Rightarrow k < 1 \quad \text{(closed)}
\end{align*}
\]

\[
\Omega_{\text{tot}} = \frac{3H^2}{8\pi G} \Rightarrow \frac{S}{\text{Planck}} \quad \text{today}
\]

\[
\Omega_{\text{tot}} = 1 \Rightarrow k = 0
\]

\[
\frac{S}{\text{Planck}} \Rightarrow \Omega_{\text{tot}} = 1
\]

\[
S\text{'s are function of time, except for $\Omega_{\text{tot}}$}
\]

\[
\Omega_{\text{tot}} = 1
\]

- **Supernova Results (slide)**

- **Age of the Universe ($\Lambda = 0$)**

\[
\frac{a^2}{\dot{a}^2} + \frac{1}{a^2} = \frac{8\pi G}{3} \rho \quad \Rightarrow \quad a^2 = \frac{8\pi G}{3} \rho a^2 - k
\]

\[
pa^2 = \frac{1}{a}
\]

\[
\frac{8\pi G}{3} \rho a^2 = \frac{8\pi G}{3H^2} \rho H^2 a^2 = \Omega_{\text{tot}} H^2 a^2
\]

\[
\Rightarrow a^2 = \Omega_{\text{tot}} H^0 a_0^2 \frac{1}{a} \quad \text{Also} \quad k = (\Omega_{\text{tot}} - 1) H^2 a^2 = (\Omega_{\text{tot}} - 1) H^2 a_0^2
\]

\[
\Rightarrow \dot{a}^2 = \Omega_{\text{tot}} H^0 a_0^2 \frac{1}{a} - \frac{1}{H^2} \left( \frac{\dot{a}}{a} \right) H^2 a^2
\]

\[
\Rightarrow \dot{a} = H_0 a_0 \left[ \Omega_{\text{tot}} a_0^2 \frac{1}{a} + 1 - 2\Omega_{\text{tot}} \right]^{1/2}
\]

\[
t = \frac{S}{H_0} \left[ \frac{1}{a_0^2} \right]^{1/2}
\]
\[ t_0 = \frac{1}{H_0} \int_0^1 \frac{d\alpha}{\left[ \frac{a_{\text{tot}}^2}{\alpha^2} + 1 - R_{\text{tot}}^2 \right]^{1/2}} \quad \text{(MATTER Dom.)} \]

Similarly

\[ t_0 = \frac{1}{H_0} \int_0^1 \frac{d\alpha}{\left[ \frac{a_{\text{tot}}^2}{\alpha^2} + 1 - R_{\text{tot}}^2 \right]^{1/2}} \quad \text{(RAD Dom.)} \]

\[ S_{\text{tot}}^0 = 1 \Rightarrow t_0 H_0 = \frac{2}{3}, \quad \frac{1}{2} \]

\[ H_0^{-1} = 9.8 \times 10^{-5} \text{ y}^{-1} \]

\[ H_{\text{to}} \]

\[ M_{\text{to}} \]

\[ R_{\text{to}} \]

Larger \( S_{\text{tot}}^0 \) \( \Rightarrow \) larger deceleration \( \Rightarrow \) faster expansion early on \( \Rightarrow \) lower age

Why \( t^{1/2} \) expands faster than \( t^{2/3} \)? we have expansion note here!
Note about $a$

\[ t = \int_0^{\infty} \frac{da}{\dot{a}}. \]

It seems to be sensitive to what $a$ is doing near zero, which we don't know:

\[ \int_0^\infty \frac{da}{a} \sim \frac{a_0}{a} = \frac{1}{H_0} \] (universe was expanding much faster than, so this should be small)

if $a \sim t^n$ with $n < 4 \Rightarrow H^{-1} = \left( \frac{m}{t} \right)^{-1} = \frac{t}{n}$

if $W(t)$
- Also, only time scale that enters in physical calculation is $H(t)$ at the epoch of interest, so this is always well defined.

**Age determination**

- Nuclear cosmochronology

- Stellar evolution (, CC, CH, CN, O-Ne, He burning)

\[ t_0 \]
Recall \[ \frac{h}{H^2 a^2} = \frac{8 \pi G}{3 H^2} f - 1 = \frac{f}{\text{far}} - 1 = \dot{a}_{\text{tot}} - 1. \]

At early times, curvature is negligible if \( f \ll a^{-3}, a^{-y} \)

\[ (H^2 + \frac{k}{a^2} = \frac{8 \pi G}{3} f) \]

\[ \Rightarrow H^2 \sim f \quad \Rightarrow \quad H^2 \sim a^{-3} \quad \Rightarrow \quad \dot{a}_{\text{tot}} \sim H^2 \sim a^{-4} \quad \Rightarrow \quad \dot{a}_{\text{tot}} \sim a^{-4} \quad \Rightarrow \quad \frac{h}{H^2 a^2} \sim \text{all} \quad a, a^2 \quad \text{as} \quad a \rightarrow 0 \]

\[ \Rightarrow \quad \dot{a}_{\text{tot}} \rightarrow 1 \quad \text{as} \quad a \rightarrow 0 \]

Today \( \dot{a}_{\text{tot}} \) is about unity.

Some \( \dot{a}_{\text{tot}} - 1 \sim k [\dot{a}^4, a^2] \)

Why is \( \dot{a}_{\text{tot}} \sim 1 \) if deviations grow like \( a^2 \) or \( a^2 \)?

Unless it is exactly unity (\( k = 0 \)) there is no convincing explanation (flatness problem)

\[ \log_{10} a \quad \text{versus} \quad \log_{10} a \quad (\text{we see it pretty close here}) \]