1. The linear evolution of the dark matter power spectrum \( P(k, \tau) \) after decoupling is given by

\[
P(k, \tau) = [D(\tau)]^2 [T(k)]^2 P_p(k),
\]

where \( D(\tau) \) is the linear growth factor [i.e. \( D(\tau_{dec}) \equiv 1 \)], \( T(k) \) is the transfer function and \( P_p(k) \) is the primordial power spectrum, e.g. \( P_p(k) = \left(\frac{4}{25}\right)\left(\frac{k}{H}\right)^4 P_R(k) \) where \( P_R(k) \sim k^{n_s - 4} \) is the curvature perturbation power spectrum generated by inflation, with \( n_s \) the scalar spectral index (\( n_s = 1 \) for a Harrison-Zel’dovich spectrum).

a) From Eq. (1) it follows that the present value of the spectral index, \( n(k) = \frac{d\ln P(k)}{d\ln k} \), is affected by the physics of inflation, and the dark matter and radiation content of the universe. Does the spectral index (at some fiducial scale) increase or decrease when,

i) The curvature of the inflationary potential is increased.
ii) The amount of dark matter is increased.
iii) The amount of radiation is increased.\(^1\)

In each case explain the physics of why the spectral index changes.

b) Although both induce changes in the spectral index, the inflationary effect on the power spectrum shape is somewhat different from that due to the energy contents. Explain what is this difference (assume that the overall normalization of the spectrum is fixed by comparison with observations).

c) The transfer function will be exponentially cutoff for scales \( k > k_{FS} \) due to free streaming. Therefore, in the linear approximation, the maximum value of \( \Delta(k) = 4\pi k^3 P(k) \) will be around \( k \sim k_{FS} \). Estimate the mass of the dark matter particle so that the first objects to form are of galactic mass, \( M \sim 10^{10} M_\odot \).

d) Suppose the dominant component of dark matter was made of neutrinos of mass \( m_\nu \sim 30 eV \) (with usual weak interaction cross section). Sketch the shape of the linear power spectrum, and calculate the relevant scales. Using your deep understanding of the physical meaning of the power spectrum, sketch a realization of the density field.

\(^1\)Just for fun, you may want to think how one can do that without affecting the expansion rate during nucleosynthesis.
2.

a) Starting from the equations of motion for dark matter perturbations (with $\Omega_m = 1$) for scales below the Hubble radius, derive the result sketched in class for the 2nd order solutions to the density and velocity divergence fields,

$$
\delta^{(2)}(k, \tau) = \int [\delta_D] F_2(k_1, k_2) \delta^{(1)}(k_1, \tau) \delta^{(1)}(k_2, \tau),
$$

$$
\theta^{(2)}(k, \tau) = -\mathcal{H} \int [\delta_D] G_2(k_1, k_2) \delta^{(1)}(k_1, \tau) \delta^{(1)}(k_2, \tau),
$$

where $[\delta_D] \equiv \delta_D(k-k_1-k_2)$, $\delta^{(1)}(k, \tau) = a(\tau)\delta_0(k)$ is the linear solution in the growing mode and,

$$
F_2(k_1, k_2) = \frac{5}{7} + \frac{k_1 \cdot k_2}{2k_1^2} \left( \frac{k_1}{k_2} + \frac{k_2}{k_1} \right) + \frac{2}{7} \left( \frac{k_1 \cdot k_2}{k_1 k_2} \right)^2,
$$

$$
G_2(k_1, k_2) = \frac{3}{7} + \frac{k_1 \cdot k_2}{2k_1^2} \left( \frac{k_1}{k_2} + \frac{k_2}{k_1} \right) + \frac{4}{7} \left( \frac{k_1 \cdot k_2}{k_1 k_2} \right)^2.
$$

b) Check that Eqs. (2-3) respect $\langle \delta \rangle = \langle \theta \rangle = 0$.

c) It is thought that at large enough scales, the galaxy density field at a given point must be a local (i.e. at the same point) function of the dark matter density field, $\delta_g = f(\delta)$. Since fluctuations are small, one can expand in Taylor series, $\delta_g = \sum_{m=0}^{\infty} b_m \delta^m$. Show that to leading order in perturbation theory, the galaxy power spectrum and skewness obey

$$
P_g(k) = b_1^2 P(k),
$$

$$
(S_3)_g = \frac{S_3}{b_1} + \frac{3b_2}{b_1^2}.
$$

Don’t forget to impose $\langle \delta_g \rangle = 0$.

d) Suppose the only non-linearities in the equations of motion were of order $p$ instead of quadratic ($p = 2$). What would be the skewness induced in 2nd order PT for $p = 3$? How would the scaling in the $p = 2$ case [$\langle \delta^n \rangle_c \sim \langle \delta^2 \rangle^{n-1}$] change if $p = 4$?