Introduction

I'm going to start with the basic ideas of GR and then in the next few classes we will discuss in some detail the main tools we need to do inflationary cosmology and perturbations, and also the things we need to contrast later with modifications of GR proposed to explain the current acceleration of the universe.

Emphasis will also be on observational signatures as well; GR is sometimes presented in very abstract form as a "beautiful theory", but all beauty is useless if not matched by observations!

In the beginning of 1905: Einstein special relativity (SR) was proposed, there was an incompatibility in physics between

i) the principle of relativity, which says that all physical laws are the same in any inertial reference system,

ii) the Galilean transformations, which give the transformation law between inertial reference systems.

iii) Maxwell's electrodynamics, which is not invariant under Galilean transformations.

Einstein solved this problem in SR, by replacing Galilean transformations with Lorentz transformations and extending Newtonian mechanics to relativistic mechanics, relevant for speeds comparable to the speed of light (the new "absolute" of physics). In this way Maxwell laws obeyed the principle of relativity.

However, a new problem arose from this great achievement. Namely, whereas the gravitational law was invariant under Galilean transformations, it was not invariant under the new Lorentz transformations! Therefore, gravitational physics did not obey the principle of relativity.
attempts to give a formulation of gravity that respects Lorentz transformations fail due to conservation of energy (or lack of gravitational redshift in SR) as we now discuss. Suppose there is a particle of mass \( m \) that can decay into 2 photons

\[
\begin{align*}
\bullet & \rightarrow h\nu \quad h\nu \\
m & \Rightarrow mc^2 = 2h\nu
\end{align*}
\]

Then we can imagine the following experiment, in which the particle decays at \( x=0 \) and is rebuilt at \( x=h \) in a gravitational field:

\[
\begin{align*}
\text{mirror} \\
x=h & \quad \text{mirror} \\
0 & \quad \text{m}
\end{align*}
\]

\[
E_0 = mc^2 < E_f = mc^2 + mgh
\]

The conclusion from this gedanken experiment is that the gravitational field must affect the energy of photons in such a way that

\[
\begin{align*}
E_0 &= mc^2 = 2h\nu = E_f = 2h\nu' + mgh \\
\Rightarrow 2h\nu(1 - \frac{gh}{c^2}) &= 2h\nu' \Rightarrow \nu' &= \nu \left(1 - \frac{gh}{c^2}\right)
\end{align*}
\]

\[
\text{This is the famous gravitational redshift on a photon as it climbs a gravitational field by distance } h. \text{ In terms of the gravitational potential } \Phi(r) = -\frac{GM}{r}, \text{ we have } \nu' = \nu \left(1 + \frac{\Phi}{c^2}\right) \text{ where we use:}
\]

\[
\frac{1}{R+h} \sim \frac{1}{R} \left(1 - \frac{h}{R}\right)
\]

Now, the point is that gravitational redshift in SR is not possible. For simplicity we consider a static gravitational field. In absence of such field, light rays propagate in straight lines at 45° angles in spacetime diagram. Although we don't know how the will propagate in
The "new" Lorentz-invariant gravity, what we know is that rays separated by \( \Delta t \) at the source \( @ z = 0 \) will be separated by the same \( \Delta t \) at the reception \( @ z = h \), since both rays are affected equally due to the (static) gravitational field:

\[
\frac{c^2 \Delta t^2}{z^2} = ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2
\]

Formally, this follows from the fact that both times are proper times (both emissions and receptions happen at the same point in space) then

\[
- c^2 \Delta t^2 = ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}
\]

From here we see that if we generalize the Minkowsky metric \( \text{M} \) to a (dynamical) more general one \( \text{G} \) that could depend on coordinates then we can explain gravitational redshift, in particular we see that \( G_{00} \) will have to be relaxed (in the weak-field approximation at least) to the gravitational potential \( \Phi \).

Another, perhaps more philosophical, issue was why do interstellar reference systems play such a privileged role in the principle of relativity - inspired by Mach, who thought that only the distribution of matter in the universe could assign a privileged reference system (corresponding to that in which total angular momentum is zero), and that moves at constant speed with respect to center of mass of the universe), Einstein proposed a general version of the principle of relativity, in which all physical laws must be formulated in equivalent way in any reference system. This is also known as the covariance of physical laws under general coordinatization.
The key aspect that allows connection between SR and the formulation of GR is the equivalence principle. Indeed, from the equality between inertial and gravitational mass it follows that a gravitational field and a accelerated frame are locally (in space and time) indistinguishable.

\[ \mathbf{a} = -\mathbf{g} \]

The equivalence is only local because by extending spatially the size of the system one can see in the gravitational field case that the forces converge to their sources; the same can be done by waiting long enough to accumulate small deviations.

This equivalence means that understanding how to incorporate the gravitational field is equivalent to understanding physical laws in accelerated (non-inertial) frames.

There are two formulations of the equivalence principle, one weak and one strong:

**Weak EP**: the trajectories of particles in free fall do not depend on their characteristics, only on initial conditions (i.e., free particles will follow geodesics of space-time). By going into free-fall, one is locally in an inertial system where SR is valid.

**Strong EP**: the effects of gravity can be eliminated by going into free fall (change of reference system), where the laws of physics take the same form as in SR.

Here we see how SR gets incorporated into GR.
Finally, the idea of Einstein’s GR is that the gravitational field is not a force, but rather is due to the curvature of space-time; therefore all particles feel curvature equally, thus incorporating the weak equivalence principle.

The curvature of space-time is given by the metric, a quantity which is determined by the distribution of matter in the universe through Einstein’s field equation, incorporating to some extent Mach’s ideas (though in vacuum there is no unique solution).

In a locally inertial system (free fall) gravity is reduced to $\mathbf{mg}$ and SR holds: the character of inertial frame is thus given by the form the metric takes.

- **Geodesics**

As we showed above, conservation of energy leads to gravitational redshift, in other words, a clock will run faster in a region of higher (less negative) gravitational potential.

One can then pose the following interesting question: how do we need to move a clock so that compared to a stationary clock it will run as fast as possible, say after 1 hour, after which we bring the moving clock back to compare it with the stationary clock?

There are two effects at work here, one is gravitational redshift that makes the clock run faster as we move it up away from the earth, the other is special relativistic time dilation that makes the clock tick slower when in motion.
So, clearly there must be an optimal way to move the clock to achieve maximum time compared to a stationary clock.

The answer, as you might expect, is that in order to have maximal time elapsed in moving clock compared to stationary clock (after this one measures, say, \( dt = \lambda \text{ hr} \)) is to shoot up the moving clock with such initial velocity so that it stays in free fall the whole time and lands back at the stationary clock at the same time.

In order to see this, we well use the metric that follows from gravitational redshift, where only \( g_{00} \) is non-trivial:

\[
\text{ds}^2 = -c^2 dt^2 \left( 1 + \frac{2\lambda}{c^2} \right) + dx^2 + dy^2 + dz^2
\]

The proper time \( \tau \) between events \( A \) and \( B \) is given by:

\[
\tau_{AB} = \int_A^B \sqrt{-\text{ds}^2} = \int_A^B \sqrt{\left( 1 + \frac{2\lambda}{c^2} \right) dt^2 - \frac{1}{c^2} \text{d}x^2} = \int_A^B \sqrt{\left( 1 + \frac{2\lambda}{c^2} \right) - \frac{1}{c^2} \frac{\text{d}x^2}{c^2}} = \int_A^B \left[ 1 - \frac{1}{c^2} \left( \frac{\text{d}x}{c} \right)^2 \right] \text{d}t
\]

where we have expanded \( \frac{2\lambda}{c^2}, \frac{\text{d}x}{c} \ll 1 \). Notice the competing effects of time dilatation and gravitational redshift, and thus maximal \( \tau_{AB} \) requires minimiza of

\[
\int_A^B \text{d}t \left( \frac{1}{2} m v^2 - m \lambda \right)
\]

which is the non-relativistic action, leading to the principle of least action!

The relativistic interpretation (in the context of a gravitational field) is that \( T - V \) is that \( T \) corresponds to time dilation and \( V \) to gravitational redshift.

Geodesics are exactly those curves we have been talking about, those followed by free (albeit any force other than gravity, which is encoded by metric) test particles. By "test particle," we mean that they do not produce gravity.

Now, in general, for a more general metric (the geodesic equation will not give just the Euler-Lagrange equation of classical Newtonian mechanics, but rather its GR generalization).

Indeed, consider a general metric so that \( ds^2 = g_{00} \text{d}t^2 + \text{d}x^2 + \text{d}y^2 + \text{d}z^2 \).
Then the equation of motions are encoded in the variational principle,

\[ \delta \int \sqrt{-g_{\alpha\beta} \, dx^\alpha dx^\beta} = 0 \]

which can be thought of emerging from a Lagrangian for a free particle,

\[ L = \sqrt{-g_{\alpha\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt}} \]

\[ \Rightarrow S = \int dt \, L \]

where \( T \) is some parameter along the geodesic (such as proper time if mass \( m \neq 0 \)). Doing the variation, after some algebra, one gets:

\[ \frac{d^2 x^\alpha}{dt^2} = -\Gamma^\alpha_{\beta\delta} \frac{dx^\beta}{dt} \frac{dx^\delta}{dt} \]

(geodesic equation)

where \( \Gamma^\alpha_{\beta\delta} \) are the Christoffel symbols, given by

\[ g_{\delta\gamma} \Gamma^\delta_{\beta\gamma} = \frac{1}{2} \left( \frac{\partial g_{\beta\delta}}{\partial x^\gamma} + \frac{\partial g_{\gamma\delta}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\delta} \right) \]

The geodesic equation can be rewritten in a more convenient form by using the 4-velocity \( u^\alpha = \frac{dx^\alpha}{dt} \) where \( u^\alpha u_\alpha = -1 \)

then:

\[ \frac{d^2 x^\alpha}{dt^2} = -\Gamma^\alpha_{\beta\delta} u^\beta u^\delta \]

One can think of geodesics as the generalization of straight lines to curved spaces. Consider the following definition of a straight line:

"the tangent to such line at one point is parallel to the tangent at a previous point"

What do we mean by parallel? What we mean is that we do a parallel transport of the tangent vector \( v^\alpha \) from \( A \) to \( B \)
and compose it to $\mathbf{v}$ at the new point B

Note that in general we can parallel transport any vector $\mathbf{u}$ along a curve defined by tangent vector $\mathbf{v}$:

for a non-straight line:

Parallel transport means that the derivative of the vector $\mathbf{u}$ does not change along the curve, in other words that the directional derivative of $\mathbf{u}$ along $\mathbf{v}$ is zero,

$$(\mathbf{v} \cdot \nabla) \mathbf{u} = 0$$

(parallel transport)

A geodesic then can be defined in geometric terms as a curve with a tangent vector that is parallel transported along itself!

$$(\mathbf{v} \cdot \nabla) \mathbf{v} = 0$$

In SR it generalizes to 4-vectors as

$$\nabla_\beta u^\alpha = 0$$

And in GR using the principle of equivalence,

$$\nabla_\beta u^\alpha = 0$$

where the covariant derivative is given by

$$\nabla_\beta u^\alpha = \partial_\beta u^\alpha + \Gamma^\alpha_{\beta\gamma} u^\gamma$$
\[ n^\beta \nabla_\beta + T^\alpha _{\mu \beta} n^\mu n^\nu = 0 \]

\[ \frac{d^2 x^\lambda}{dt^2} = \frac{d^2 x^\alpha}{dt^2} \quad \Rightarrow \quad \frac{d^2 x^\lambda}{dt^2} = -T^\lambda _{\mu \beta} n^\mu n^\nu \]

which is the expression that follows from the Lagrangian as well, as earlier.

Now, according to the principle of equivalence, there is a freely falling frame in which

\[ g_{\mu \nu} = \eta_{\mu \nu} \]

\[ \eta_{\mu \nu} < 0 \]

Then \[ \frac{d^2 x^\lambda}{dt^2} = 0 \quad \Rightarrow \quad \text{straight line, as expected in SR} \]

Notice that this is not just true at a point, but along a geodesic. A freely falling frame is a local inertial frame along a geodesic. Although one can define a set of coordinates with axes perpendicular to geodesic, the property above will still hold at geodesics, not at nearby points.

A locally inertial frame at a point is found by doing a coordinate transformation so that \[ g_{\mu \nu} = \eta_{\mu \nu} \] and \[ T^\mu _{\nu \rho} = 0 \]

Defining \[ x' = x - x^\mu \frac{\partial x^\mu}{\partial x'^\mu} x^\alpha x^\beta \]

and \[ g'_{\mu \nu} = g_{\mu \nu} \frac{\partial x^\mu}{\partial x'^\mu} \frac{\partial x^\nu}{\partial x'^\nu} \]

one can check \[ g'_{\mu \nu} = \eta_{\mu \nu} \] and \[ T'_{\mu \nu \rho} = 0 \].