The Action of the Gravitational Field

Modern theories of gravity are easiest described in terms of their action, so we first should recall the action of the gravitational field in 6R. Let's see that the Einstein-Hilbert action,

$$ S_0 = \frac{1}{16\pi G} \int \sqrt{g} \ R \ d^4x = \frac{M_p^2}{2} \int \sqrt{g} \ R \ d^4x $$

gives rise to Einstein equations. The Ricci curvature scalar is \( R = g^{\mu\nu} R_{\mu\nu} \) thus a variation \( \delta g_{\mu\nu} \) in the metric gives a change in the integrand

$$ \delta (\sqrt{g} R) = \sqrt{g} R_{\mu\nu} \delta g^{\mu\nu} + R \delta g + \sqrt{g} g^{\mu\nu} \delta R_{\mu\nu} $$

Now, we have \( \delta R_{\mu\nu} = (\delta \Gamma^\lambda_{\mu
u}) \nu - (\delta \Gamma^\lambda_{\nu\mu}) \mu ; \lambda \)

$$ \Rightarrow \sqrt{g} g^{\mu\nu} \delta R_{\mu\nu} = \sqrt{g} \left[ (g^{\nu\lambda} \delta \Gamma^\mu_{\lambda\nu}) \nu - (g^{\mu\lambda} \delta \Gamma^\nu_{\lambda\mu}) ; \lambda \right] $$

$$ = \frac{\partial}{\partial x^\lambda} \left[ \sqrt{g} g^{\nu\lambda} \delta \Gamma^\mu_{\lambda\nu} \right] - \partial_{\lambda} \left[ \sqrt{g} g^{\mu\lambda} \delta \Gamma^\nu_{\lambda\mu} \right] $$

$$ \therefore \delta R_{\mu\nu} = \frac{\partial}{\partial x^\lambda} \left( \sqrt{g} g^{\nu\lambda} \delta \Gamma^\mu_{\lambda\nu} \right) - \partial_{\lambda} \left[ \sqrt{g} g^{\mu\lambda} \delta \Gamma^\nu_{\lambda\mu} \right] $$

Thus, this is a total divergence and drops out when we integrate over all space. We are only left with two terms, the second can be written as

$$ \delta \sqrt{g} = \frac{1}{2} \sqrt{g} \ \delta g = \frac{1}{2} \sqrt{g} \ \delta \text{log} $$

Now \( \delta \text{log} = \text{ln} (g + \delta g) - \text{log} = \text{ln} \ \frac{g + \delta g}{g} = \text{ln} \ \text{det} \left[ g^{\mu\nu} (g_{\mu\nu} + \delta g_{\mu\nu}) \right] $$

$$ = \text{ln} \ \text{det} (g^{\mu\nu} + g^{\mu\nu} \delta g_{\mu\nu}) \approx \text{ln} \ (1 + g^{\mu\nu} \delta g_{\mu\nu}) \approx g^{\mu\nu} \delta g_{\mu\nu} $$

$$ \Rightarrow \delta \sqrt{g} = \frac{1}{2} \sqrt{g} \ g^{\mu\nu} \delta g_{\mu\nu} $$

To bring everything to the same form, note first term can be rewritten as

$$ R_{\mu\nu} \delta g^{\mu\nu} = - R_{\mu\nu} g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta} = - R^{\alpha\beta} \delta g_{\alpha\beta} $$

$$ g^{\mu\nu} g_{\mu\nu} = \delta^\alpha_\mu $$
Then we have,

$$\delta S_0 = -\frac{\Lambda g^2}{2} \int \sqrt{g} \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \delta g_{\mu\nu} \, dx$$

Thus the action being stationary simplifies the vacuum Einstein equations,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

In order to introduce sources we need to add the action for matter fields, with

$$\delta S_m = \frac{1}{2} \int \sqrt{g} \, T_{\mu\nu} \delta g_{\mu\nu} \, dx$$

i.e. $T_{\mu\nu} \equiv \frac{2}{V g} \frac{\delta S_m}{\delta g_{\mu\nu}}$

which generalizes the standard definition in flat space. One says that "gravity couples to the energy-momentum tensor.

Thus here it is obvious that usual field equations follow:

$$\delta S_m = \frac{1}{2} \int \sqrt{g} \, T_{\mu\nu} \delta g_{\mu\nu} \, dx$$

$$\Rightarrow \frac{\delta S_0 + \delta S_m}{\delta g_{\mu\nu}} = 0 \Rightarrow R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{\Lambda g^2}{2} T_{\mu\nu} = 8\pi G T_{\mu\nu}$$

The weak field limit action

The Einstein-Hilbert action looks very different from standard field theories; however, it can be brought to the same structure in the weak field limit where we expand in small $g_{\mu\nu}$ where

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

i.e. have an deviation from flat space time, describing gravitons. Symbolically, since $R$ involves two derivatives, we have something like

$$S_0 = \frac{\Lambda g^2}{2} \int \delta^4 x \left( \phi \phi + \phi \partial^2 \phi + \frac{1}{2} \phi \partial^2 \phi + \ldots \right)$$

The first term is no different than the standard $\phi^4$ contribution to the scalar field action or $\delta A \delta A$ for photons; whereas the higher
order terms describe the interactions of the graviton with itself (due to non-linearities in the Einstein equations), in the language of field theory gravitons themselves carry energy-momentum and thus they couple to gravity, i.e. itself. The matter action reads

\[ \frac{1}{2} \int d^4x \; h_{\mu}^\nu \; T^{\mu\nu} \]

the graviton part will not contribute to the equation of motion. We can write symbolically everything as

\[ S = \int d^4x \left[ \frac{M_0}{2} \left( \partial h^\mu \partial h_\mu + h_\mu \partial h_\mu + h_\mu \partial h_\mu + \ldots \right) \right] \]

to rescale the graviton field as \( h_{\mu}^\nu \rightarrow M_0 h_{\mu}^\nu \) we can leaving the action in form similar\( \) to a scalar field for the quadratic piece.

\[ S = \int d^4x \left[ \partial \tilde{\phi} \partial \tilde{\phi} + \frac{M_0}{2} \left( \partial \tilde{\phi} \partial \tilde{\phi} + \partial \tilde{\phi} \partial \tilde{\phi} + \ldots + \partial \tilde{\phi} \partial \tilde{\phi} \right) \right] \]

we see from this redefinition trivially that the graviton couples to itself and other fields through the coupling strength \( \frac{M_0}{2} \), the smallness of this parameter (say compared to other forces\( ) \) is a manifestation of the weakness of gravitational forces.

Doing the proper calculation up to order \( h^2 \) one recovers the weak-field action \( S_{WF} \):

\[ S_{WF} = \int d^4x \left( \frac{M_0^2}{4} J_{\mu\nu} + \frac{1}{2} \partial J_{\mu\nu} T^{\mu\nu} \right) \]

where:

\[ J_{\mu\nu} = \frac{1}{2} \partial \partial h_{\mu\nu} - \frac{1}{2} \partial \partial h_{\mu\nu} h_{\mu\nu} - \partial \partial h_{\mu\nu} \partial \partial h_{\mu\nu} + \partial \partial h_{\mu\nu} \partial \partial h_{\mu\nu} \]

We can use this to compute the static potential between two sources, for example. Before we do that, let's recall some basics of scalar field theory.
Consider a free scalar field with lagrangian

\[ \mathcal{L} = \frac{1}{2} \left[ (\partial \phi)^2 - m^2 \phi^2 \right] \]

The field obeys the Klein-Gordon equation \((\partial^2 + m^2) \phi = 0\)
which can be solved in terms of plane waves \(\phi(x) = e^{i(k \cdot x - \omega t)}\) with dispersion relation \(\omega^2 = k^2 + m^2\).

The propagator, or Fourier-Green's function, obeys

\[(\partial^2 + m^2) D(x-y) = -\delta^4(x-y)\]

which can be solved by Fourier transforms,

\[ D(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{i k (x-y)}}{k^2 + m^2 + i\varepsilon} \]

Physically, \(D(x)\) describes the amplitude for a perturbation in the field to propagate from the origin to \(x\).

In field theory, forces, or interactions, are mediated by the interchange of virtual particles. If we have a "source" \(J_1\), then nearby a disturbance of the field to a "sink" \(J_2\), and the interaction potential is characterized by the propagator through,

\[ W(J) = -\frac{1}{2} \int \int d^4x \, d^4y \, J(x) \, D(x-y) \, J(y) \]

or, in Fourier space,

\[ W(J) = -\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \frac{J^*_k}{k^2 + m^2 + i\varepsilon} \]

where \(J^*_k = J^T_k\) for a real source \(J(x)\). If \(J = J_1 + J_2\), the interaction part is through terms like

\[ W(J) = -\frac{1}{2} x^2 \int d^4k \, J^*_2(-k) \frac{1}{k^2 + m^2 + i\varepsilon} J_1(k) \]
the rest being self-interactions. Consider a source $J_1$ localized at $\mathbf{x}_1$ and $J_2$ localized at $\mathbf{x}_2$, thus

\[ J_1(x) = \delta_{(3)}^i(x - x_1) \quad J_2(x) = \delta_{(3)}^i(x - x_2) \]

\[ \Rightarrow W_{\text{int}} = -\int dx^0 \int dy^0 \int \frac{dk_0}{2\pi} e^{ik_0(x-y)} \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{k} \cdot (x-x_2)}}{k^2 + m^2} \]

the integral over $y^0$ sets $k^0 = 0 \Rightarrow k^2 = -k^2$ \quad (+---)

then

\[ W_{\text{int}} &= \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{k} \cdot (x-x_2)}}{k^2 + m^2} \quad \Rightarrow \quad \Delta T \quad V_{\text{int}}(r) \]

\[ \Rightarrow V_{\text{int}}(r) = -\int \frac{d^3k}{(2\pi)^3} \frac{e^{-mr}}{k^2 + m^2} = -\frac{e^{-mr}}{4\pi r} \]

Notice that the potential is negative, i.e. attractive, so sources of the same sign attract. Also notice that we got a $1/r^2$ potential because we are in 3D (at 4) and the Lagrangian density involves two powers of the spacetime derivative (since any term involving only one derivative, e.g. $y^0 \phi$ is not Lorentz invariant).

BTW, you can do the same calculation for a vector field like the photon, instead of a scalar, and you'll see that you recover a different sign for the potential, i.e. interaction between charges of the same sign is repulsive, like it should be for electromagnetism.

This happens because the contraction of sources $J^\mu J_\mu$ gives an additional minus sign for spatial charges. For tensor fields such as gravity, as we derive below, the same thing happens but there are two indices contracted so we get no change in sign, i.e. the force is attractive between same sign charges, as in the scalar case.
Now, let's go back to the gravitational weak-field action. After some serious algebra, using the harmonic gauge condition \( \partial_\mu h^\mu = \frac{1}{2} \partial_\mu h^\mu \), we can write

\[
S^{\text{grav}} = \frac{1}{2} \int d^4x \left[ \frac{M_p^2}{4} \left( \partial_\mu h^{\mu \nu} \partial_\nu h^{\alpha \beta} - \frac{1}{2} \partial_\mu h^{\mu \nu} \partial_\nu h^{\alpha \beta} \right) + h^{\mu \nu} T^{\mu \nu} \right] h^{\mu \nu} \]

whose gravity only part can be rewritten as

\[
\frac{M_p^2}{4} \int d^4x \left[ -h^{\mu \nu} \kappa_{\mu \nu} ; J_0 \right. \left. (\Delta^2) h^{\alpha \beta} + \Theta (h^3) \right]
\]

therefore the propagator requires to invert the matrix

\[
\kappa_{\mu \nu} ; J_0 \equiv \frac{1}{2} \left( \eta_{\mu \lambda} \eta_{\nu \sigma} + \eta_{\mu \sigma} \eta_{\nu \lambda} - \eta_{\mu \nu} \eta_{\lambda \sigma} \right)
\]

thinking of \( \mu \nu \) and \( J_0 \) as just two indices. In fact \( \kappa = \kappa^{-1} \), i.e. you can check \( \kappa_{\mu \nu} ; J_0 \kappa^{\mu \nu} ; J_0 = I_{\mu \nu} \) where the identity matrix is

\[
I_{\mu \nu} ; J_0 \equiv \frac{1}{2} \left( \eta_{\mu \lambda} \eta_{\nu \sigma} + \eta_{\mu \sigma} \eta_{\nu \lambda} \right)
\]

Thus, the graviton propagator in the harmonic gauge is

\[
D_{\mu \nu} (k) = \frac{1}{2} \left( \eta_{\mu \lambda} \eta_{\nu \sigma} + \eta_{\mu \sigma} \eta_{\nu \lambda} \right) \frac{1}{k^2 + \lambda^2}
\]

Consider now two particles with stress energy \( T^{\mu \nu}_{(1)} \) and \( T^{\mu \nu}_{(2)} \) interacting via graviton exchange. The scattering amplitude is then proportional to

\[
G T^{\mu \nu}_{(1)} D_{\mu \nu} (k) T^{\alpha \beta}_{(2)} = \frac{G}{2k^2} \left( 2T^{\mu \nu}_{(1)} T^{\alpha \beta}_{(2)} - T^{\mu \nu}_{(1)} T^{\alpha \beta}_{(1)} - T^{\mu \nu}_{(2)} T^{\alpha \beta}_{(1)} \right)
\]

for non-relativistic sources, \( t^{\mu \mu} \) is the dominant component, thus

\[
\rightarrow \frac{G}{2k^2} t^{\mu \mu}_{(1)} t^{\mu \mu}_{(2)}
\]

and thus the interaction potential gives

\[
G \int d^3r \ t^{\mu \mu}_{(1)} (x) \int d^3r \ t^{\mu \mu}_{(2)} (y) \int d^3k \ e^{-i(k \cdot (x-y))} \frac{1}{k^2}
\]

so 1-graviton exchange gives Newtonian gravity.