Introduction

Historically, the first interesting approach involving extra dimensions was by Kaluza and Klein (KK) who showed that by adding an extra dimension one can unify electromagnetism and gravity — let’s see how this idea works in a toy example first, and how extensions of this idea impact modern theories of QFT.

We start from a simple example, a real scalar field in 4+1 dimensions with the extra (5th) dimension (spatial) compactified with compactification scale $L$, which we will take to be a radius of size $L$. Clearly, one expects that at scales large compared to $L$, the extra dimension will not be noticeable, only by probing short-distance scales, compared to $L$, one can realize there is an extra dimension.

Start from the Lagrangian density (metric signature line is $-+++$)

$$\mathcal{L} = -\frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi$$

where $x^\mu = 0,1,2,3$ are the usual 4D coordinates, and $x^5 = y$ is the extra dimension. The scalar field $\phi(t, x^\mu, y)$ is periodic in $y$ due to compactification on a circle,

$$\phi(x^\mu, y) = \phi(x^\mu, y + 2\pi L)$$

Then it admits the Fourier representation,

$$\phi(x^\mu, y) = \sum_{n=-\infty}^{\infty} \phi_n(x^\mu) e^{in y/L}$$

and so we can write

$$\mathcal{L} = -\frac{1}{2} \sum_{n,m} \left( \partial_{\mu} \phi_n \partial^{\mu} \phi_m - \frac{\delta_{nm}}{L^2} \partial_{\mu} \phi_n \partial_{\mu} \phi_m \right) e^{i (mn) y/L}$$

and the action (we are assuming Minkowski space for simplicity here)

$$S = \int d^4x \int_0^{2\pi L} dy \mathcal{L} = -\frac{2\pi L}{2} \int d^4x \sum_{n=-\infty}^{\infty} \left( \partial_{\mu} \phi_n \partial^{\mu} \phi_n + \frac{n^2}{L^2} \phi_n \partial_{\mu} \partial^{\mu} \phi_n \right)$$
where we have integrated over $y$ and used the reality condition
$
\Phi_m(x) = \Phi_m^*(x)$. We see that tracing over the 5th dimension
generated an effective 4D theory of an infinite number of
4D fields $\phi_m(x)$ - Introducing the normalized fields

$$\phi_n = \sqrt{2/m} \, \Phi_m$$

The action is:

$$S = \int d^4x \left( -\frac{1}{2} \partial_m \phi_0 \partial^m \phi_0 \right) - \int d^4x \sum_{n=1}^{\infty} \left( \partial_m \phi_n \partial^m \phi_n^* + \frac{k^2}{L^2} \phi_n \phi_n^* \right)$$

Thus, the spectrum of the compactified theory consists of, from the
4D point of view, of a single real scalar field known as the
"zero-mode" $\phi_0$, and an infinite number of massive complex
scalar fields with masses proportional to the inverse of the
Compactification scale, $m_n = k/L$, these are the so-called
KK modes - at low energy, at large scales, $E \ll L^{-1}$ only the
zero mode is important, when $E \sim L^{-1}$ the KK modes
become relevant.

Now consider the KK proposal for a 5D action,

$$S = \frac{M_5^3}{2} \int d^5x \, d\bar{y} \sqrt{G} \, R_5$$

where $M_5$ is the 5D Planck mass, $G$ and $R_5$ are 5D metric quantiti-
Again the $y$ coordinate is periodic with period $2\pi L$, then we
expand

$$\Theta_{AB} (x^a, y) = \sum_{m=-\infty}^{\infty} \Theta_{AB} (x^a) \, e^{im\pi L}$$

the calculation is considerably more complex, but the result for the
zero mode is rather simple - Introducing the zero mode definition

$$\Theta^{(0)}_{AB} = e^{iN/3} \left( G_{\mu\nu} + e^{iN/3} A_\mu A^\nu \right)$$

and setting $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$G^{(0)}_{55} = -e^{iN/3} A_\mu$$

the solution is

$$G^{(0)}_{55} = -e^{iN/3} A_\mu$$

as usual.
The zero-mode action needs

\[ S_0 = M_\ast^2 ITL \int d^4x \sqrt{g} \left[ R_4(g) - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} \phi^2 e^{-\frac{2\phi}{g}} f_{\mu
u} f^{\mu\nu} \right] \]

Comparing with the usual 4D gravity action, we see that \(M_\ast^2 = \frac{M_5^2}{2} L\), and therefore Newton’s constant is related to the 5D Planck mass and compactification scale by

\[ G_N = \frac{1}{16 \pi^2 M_5^3 L} \]

The result above is that electromagnetic and gravity fields in 4D may have a common origin in a 5D gravitational field. In addition to the zero mode there are massive gravitons with mass \(m_5^2 = k^2 / L^2\) at the \(k^4 / L^4\) level. Note that the number of degrees of freedom would rise, the zero mode has a total of \(5\) dof: \(2\) for a massive 4D graviton, \(1\) for a scalar and \(2\) for a vector field. This is the same as a 5D massive graviton: \(4\) spatial components lead to \(5 \times 4 = 20\) components and traceless plus transverse conditions take out \(14 = 5\) components, leaving \(5\) physical components (gauge freedom plays no role, as in 4D case, simple tensor perturbations are gauge invariant).

"Brane words"

Modern ideas for modifications of gravity are based on the notion that our 3+1 world could be a 3D surface in a higher dimensional space, known as "brane". The idea is that standard massive particles are localized in the brane, and that only gravity can "see" the extra dimensions. Gravity gets modified at large or small scales depending on whether the extra dimensions are infinite or compactified to a small volume.

In the ADD model (Arkani-Hamed, Dimopoulos & Dvali, hep-ph/9803315)

there are \(N\) extra dimensions compactified at equal size \(L\), and the gravitational action reads

\[ S_{ADD} = \frac{M_\ast^{2+N}}{2} \int d^3x \int_0^{2\pi L} d^N y \sqrt{G} R_{4+N} \]
which leads to the effective 4D action for gravitino
\[ \frac{M_{*}^{2+N}}{2} (2\pi L)^{N} \int d^{4}x \sqrt{g} R \]

this leads to \( M_{Pl}^{2} = M_{*}^{2+N} (2\pi L)^{N} \) - If we postulate that the quantum gravity scale is at \( M_{*} \sim \text{TeV} \) (remember \( M_{Pl} \) here is a defined quantity, not a fundamental one like \( M_{*} \)) then the size of extra dimensions is given by

\[ L \sim 10^{-17} \frac{M_{Pl}}{N} \text{ cm} \]

For one extra dimension, \( L \sim 10^{13} \text{ cm} \), which is ruled out, because we probe distance scales smaller than this and gravity is still 4D gravity; for \( N=2 \) \( L \sim 10^{-2} \text{ cm} \) it is the interesting case since it predicts deviations of the potential at sub-millimeter scales, which is being probed by current experiments.

The potential between two non-relativistic sources is in this scenario,

\[ V(r) = -\frac{G_{N} m_{1} m_{2}}{r} \sum_{n=-\infty}^{\infty} (\psi_{n}(y=0))^{2} e^{m_{n}r} \]

where \( \psi_{n}(y=0) \) denotes the wave function of the \( n \)th kink mode of the brane and \( m_{n} = M_{Pl}/L \) is the mass of the \( n \)th kink mode - At large scales one replaces,

\[ V(r) \sim -\frac{G_{N} m_{1} m_{2}}{r} \ 	ext{r >> L} \]

whereas in the opposite limit \( r \ll L \) the law for \((4+N)\)-dimensional gravity is obtained,

\[ V(r) \sim -\frac{m_{1} m_{2}}{H_{*}^{2}} \frac{1}{(M_{*} r)^{N}} = -\frac{m_{1} m_{2}}{M_{*}^{2}} \left( \frac{2\pi L N}{r} \right) \]

On the other side of the spectrum of modifications of gravity is the case where the extra dimensions are infinite in extent, leading to large-scale modification of gravity and new solutions for the late time evolution of the universe, leading to acceleration.

A model proposed by DGP (Deffayet-Capdekarz & Polchinski, hep-th/0005026) has the following gravitational action (with one extra dimension)
Again, it is assumed that there lives an odd dimensional 3D brane.

The possible origin for the second term in the action is that the interaction of D-branons and matter in the brane can generate the 4-D Ricci tensor through quantum corrections.

To look at asymptotic behavior, it is convenient to obtain a length scale

\[ \ell = \frac{\Lambda_p}{\Lambda_5} \]

we see that as \( \ell \to \infty \), we recover 4D gravity, for \( \ell \geq \ell_c \) therefore we expect GR and for \( \ell < \ell_c \) one expects 5D behavior instead.

In detail, things are a bit more tricky than this naive expectation.

Let's first consider a scalar analog of the D6P action to look at the propagator:

\[ S = \Lambda_5^2 \int d^4x dy \partial \phi \partial \phi + \Lambda_6^2 \int d^4x dy \partial \phi(x) \partial \phi(y) \partial \phi(y) \partial \phi(x) \]

the equation of motion for the propagator reads,

\[ (\Lambda_5^2 \partial^2 + \Lambda_6^2 \partial^2) \phi = \delta^4(x - y) \]

in Fourier space this corresponds to (imposing approximate \( \Lambda_5 \sim \Lambda_6 \))

\[ \hat{\phi}(p, y) = \frac{1}{\Lambda_6^2 p^2 + 2m^2} e^{-p^2 y} \]

which leads to the potential

\[ V(r) = -\frac{1}{8 \pi M^2} \left[ \frac{\pi}{r} \left( \sin \left( \frac{r}{r_0} \right) \frac{1}{2} \right) \right] + \frac{1}{2} \cos \left( \frac{r}{r_0} \right) \left[ 1 - 2 \sin \left( \frac{r}{r_0} \right) \right] \]

with \( G(\alpha) = \alpha + \ln \alpha + \int_0^\alpha \frac{\sin^{-1} x}{x} \, dx \), \( Si(\alpha) = \int_0^\alpha \frac{\sin x}{x} \, dx \) and \( \alpha = 0 \) or \( \alpha + 2 \pi \)

\[ r_0 = \frac{\Lambda_6^2}{2 \Lambda_5^2} \]

Then, \( r < r_0 \) : \( V(r) \sim -\frac{1}{8 \pi M^2} \left[ \frac{\pi}{r} \right] + \frac{1}{2} \left( 1 + \ln \frac{r}{r_0} \right) \frac{\pi}{r_0} \frac{1}{2} \]

so at short distances, \( r < r_0 \) one recovers Newtonian potential.
\[ V(r) \approx -\frac{1}{8\pi^2 M^2} \left( \frac{r_0}{r} + \left( \frac{r_0}{r} \right)^2 \right) \]

so at large scales one recovers 5D gravitational law. This makes sense. Going back to the tensor case, what is more tricky is the tensorial structure of the graviton propagator. Last class we saw that for \(6k \), the tensorial structure is given by

\[ \varkappa_{\mu \nu; i \sigma} = \frac{1}{2} \left( \eta_{\lambda \mu} \eta_{\nu \sigma} + \eta_{\mu \sigma} \eta_{\nu \lambda} - \eta_{\mu \nu} \eta_{\lambda \sigma} \right) \]

However the calculation along standard lines leads to

\[ \varkappa_{\mu \nu; i \sigma} = \frac{1}{2} \left( \eta_{\lambda \mu} \eta_{\nu \sigma} + \eta_{\mu \sigma} \eta_{\nu \lambda} - \frac{2}{3} \eta_{\mu \nu} \eta_{\lambda \sigma} \right) \]

this is the same tensorial structure as that for a massive graviton, one can take the limit \(m \to 0\) and get the same as massless case with the \(2/3\) factor, thus the limit is discontinuous. Physically the problem in this case is that a massive \(5D\) graviton has 5 degrees of freedom, i.e. in \(m \to 0\) case one has 2 + 1 dof instead of 2.

As the \(m \to 0\) limit is taken 2 dof disappear because they decouple due to \(k^2 \eta_{\mu \nu} = 0\), those are ±1 helicity states. There is the 0 helicity state though that does not decouple in the \(m \to 0\) limit.

From the observational point of view one can distinguish between different tensorial structures since we measure gravity for both nonrelativistic and relativistic (e.g. photon deflection by \(m \)) sources. As we saw last time, for nonrelativistic sources we would get

\[ \frac{\Theta}{2k^2} \left( T^{00}_{(1)} T^{00}_{(0)} \right) \text{ vs. } \frac{G}{3} \frac{m^2}{2k^2} \left( T^{00}_{(1)} T^{00}_{(0)} \right) \]

where \(G\) is the Newton constant in massive graviton case (\(m \to 0\)).

It this was the only thing we knew, then one can set \(G = \frac{3}{8}\) and there is no way of telling the difference. However,
for relativistic sources we have, e.g. for a photon where $\nu = 0$

$$\frac{E}{2\hbar^2} 2T^\nu_{\mu
u} T^\nu_{\mu
u} \quad \text{vs.} \quad \frac{GM}{2\hbar^2} 2T^\nu_{\mu
u} T^\nu_{\mu
u}$$

So both things cannot be matched at the same time, e.g. if $E_M = 3.6$ to reproduce Newtonian gravity, then light deflection will be $\frac{3}{4}$ of the observed, which is ruled out.

This sounds like nonsense, how can one distinguish observationally between $m \to \infty$ and massive core with $m \to 0$? The solution of
this appears to be as follows: the one-graviton exchange is not
a good approximation as $m \to 0$, some higher-order contributions
become as important - when the theory is renormalized, then
it is expected that the $m \to 0$ limit will recover Einstein,
at least for scale smaller than the so-called "Vainshtein radius"

$$\nu = \left( \frac{6M}{M_\odot} \right)^{1/5}$$

For a gravitationally body of mass $M$.

For the DGP core something similar is expected to happen, though there
is not a proof (for the massive core also there is no proof it is sort of a
conjecture). Recently, Schaden argue that the problem does not
appear in the first place, and may have to do with expectation
regarding the fact that the breakdown of perturbation theory
$m \to 0$ is an artifact of the weak-field expansion. See hep-th/0403161
Stay tuned for future developments.

From the point of view of cosmology, the most interesting thing
about DGP gravity is that it leads to a modification of the
Friedmann equation that gives rise to accelerated expansion
and late times. This can be expected in some sense since when one modifies
LHS, it will be a Non-linear equation for $H$ that may admit $H = \text{const.}$ solution for
$f = 0$