Merger of Halos

Last class we saw how one can estimate the number density of unvirialized objects ("halos") of mass \( M \) by using the properties of random Gaussian fields smoothed on scale \( R \).

The idea is that one associates the fraction of mass in objects with mass \( > M \) with the fraction of trajectories which are "absorbed" by the barrier at \( \xi_c \) for smoothing scales \( > R \). (Note that by saying "absorbed" we automatically take care of the two contributions to the mass function.) The fraction \( f \) of objects between mass \( M \) and \( M + dM \) is given by the probability that a trajectory will have its first upcrossing through the threshold in the interval \( \xi_c^2 \) to \( \xi_c^2 + d\xi_c^2 \), i.e. we saw last class that

\[
\frac{f(M, \xi_c)}{dM} = \frac{\partial}{\partial M} \left( \frac{dM}{\partial^2} \right) = \frac{\nabla P(\xi)}{\partial M} \frac{dM}{\partial^2}
\]

where now we are thinking in terms of the variable \( \sigma^2 \) rather than \( 1/\xi_c \), since they are monotonically related we can use either of them. In terms of this variable, let \( 5 = \sigma^2 \), we can draw the random walk as:

\[
\frac{5(X)}{R} \downarrow \Delta
\]

since as we accept new Fourier modes (\( R \) decreases)

\[
\sigma^2 \downarrow \sigma^2 \uparrow \sigma^2
\]

Now, last time we thought of \( \xi_c \) as fixed (say, at 1.646) and the density field will be scaled up as time goes on according to linear...
very. For the purpose of drawing the process of merging it is easier to consider the amplitude of the field fixed, and the barrier dropping as time goes on: \( \delta_C \rightarrow \delta_C(\tau) \propto \delta_C / D_\tau(\tau) \). Then as the barrier drops (time evolution) we see that a point belongs to a halo mass \( M \) that continuously increases (sometimes by jumps: mergers, sometimes somewhat more continuously depending on resolution \( \delta_C \): accretion):

![Graph of \( \delta_C(\tau) \) vs. time](image)

Time evolution corresponds to starting at large \( \delta_C(\tau) \) (small growth factor) and then large \( S = 0^m \) (corresponding to low mass) (at high redshift only small objects had time to form as the non-linear scale is very small). As time goes on the mass increases, note however that there are discrete "large" jumps in mass (denoted by the dashed lines) corresponding to merger events, e.g. from \( M_1 \) to \( M_2 \) (merger of \( M_1 \) with \( M_2-M_4 \)) or \( M_2 \) to \( M_3 \) (big merger in this picture), etc. Recall that for each \( \delta_C(\tau) \) the halo mass is assumed to be that corresponding to the trajectory that crosses at the largest mass (smallest \( S \)), therefore the halo mass follows \( S(\delta) \) when it goes up but makes horizontal jumps when \( \delta_C(\tau) \) is decreasing.

Clearly we can consider merging of halos as a two barrier problem in which we consider the barriers at two different redshifts \( \delta_C^{(0)} \) and \( \delta_C^{(1)} \) to figure out in which halo mass our point is. Consider a subset of trajectories which make their first upcrossings at \( \delta_C^{(0)} \) for \( \delta_C^{(1)} \) and cross a second barrier of height \( \delta_C^{(1)} \) at
These trajectories represent halos that at time given by $c^{(1)}$ have various masses given by $S_1$'s and by time $c^{(2)}$ become a halo of mass given by $S_2 = \Phi^2(M_2)$. The conditional probability that one of these trajectories will make a first upcrossing of $c^{(2)}$ between $S_1$ and $S_1 + dS_1$ can be obtained from the solution for $f(M, \Phi)$ before by noting that we are solving the same problem but with the "source" of trajectories moved to $S_2$ and $c^{(2)}$, thus

$$f(S_2, c^{(2)} / S_1, c^{(1)}) = \frac{c^{(2)} - c^{(1)}}{(S_2 - S_2)^{3/2}} e^{-\frac{(c^{(1)} - c^{(2)})^2}{2(S_1 - S_2)^2}}$$

where we have used the fact that steps of the random walk are independent, so there is "translational invariance" in $S$ (and $\Phi$ since $\Phi^2$ always has the same height to be covered). What we want, however, is the opposite situation, what is the conditional probability for a first upcrossing at $S_2$ of $c^{(2)}$ given that we had a first upcrossing at $S_1$ of $c^{(1)}$ i.e. that we have a "merged" halo at a later time $c^{(2)}$ of mass $M_2$ given we have how "now" one of mass $M_1$. We can get this using the Bayes theorem

$$P(x|y) P(y) = P(x,y) = P(y|x) P(x)$$

$$\Rightarrow f(S_2, c^{(2)} / S_1, c^{(1)}) = \frac{f(S_1, c^{(1)} / S_1, c^{(1)}) f(S_2, c^{(2)})}{f(S_1, c^{(1)})}$$
This gives the conditional probability that a halo of mass $M_2$ at present time $t_1$ will merge later and by time $t_2 > t_1$ have mass between $M_2$ and $M_2 + dM_2$. Taking the limit $t_2 \to t_1$ ($\delta_2 \to \delta_1 \approx \delta_c$) we determine the transition rate:

$$f(\delta_1, \delta_2 | \delta_1, \delta_2) = \frac{1}{\sqrt{2\pi}} \left( \frac{\delta_1}{\delta_2} \right)^{3/2} e^{-\frac{1}{2} \left( \frac{\delta_1}{\delta_2} \right)^2}$$

To convert to physical units we change "time" variable from $\delta_c$ to $t$, and also from $\delta_1$ to $\delta_1 \approx \delta_c$ to $M_2 / M_c$:

$$\text{Merger rate} = \left| \frac{d \delta_1}{d t} \right| \Delta M = \frac{1}{\sqrt{2\pi}} \left( \frac{\delta_1}{\delta_c (\delta_1 - \delta_c)} \right)^{3/2} e^{-\frac{\delta_c^2}{2} \left( \frac{\delta_1}{\delta_c} \right)^2}$$

while we multiplied by $\Delta M = M_2 - M_1$ to properly normalize according to the mass is being added. The fractional mass accreted per Hubble time is $\mathcal{M} \times t \times \frac{\Delta M}{M_1}$ and looks like:

![Diagram showing the mass accretion rate with different curves for different mass ratios $M_1/M_2$. Each curve is for a fixed $M_1$. Small halos accrete a lot, while low accretion rate, mainly lots of small halos are accreted.](image-url)
The framework of the excursion set gives a nice way of estimating probabilities for different processes, e.g. how big a halo of mass $M_1$ at $t_1$ will be at $t_2$, etc. In practice one would like more detail than this, in particular, for use in galaxy formation prescriptions we would like to know the particular history of a given halo, this is what is known as the "merger tree". A merger tree is a particular realization of the random walk - of course one could start with $\delta$ and smooth it and go through the process we described, but this turns out to be unnecessary; there are more direct ways of generating merger trees. We now briefly discuss this "art".

Consider a halo of mass $M$ at time $t_0$, i.e. when the collapse threshold is given by $\delta_c (t) = \delta_c = \frac{\delta_c}{D_4(t_0)}$ (we assume the density field is linearly stable pointed to $\infty$ when $D_4 = 1$). The fraction of the mass of this halo that was in halos of mass $m$ at an earlier time $t_1 > t_0$ is given by the conditional mass function:

$$f(M, \delta_1 / M, \delta_0) = \frac{\delta_1 - \delta_0}{\left( \frac{\Omega_m}{0.7} \right)^2 - 0.7^2} e^{-\frac{(\delta_1 - \delta_0)^2}{2(\Omega_m^2 - 0.7^2)}}$$

and thus the mean number of halos of mass $m$ at $t_0$ is given by $(M/m)$ times this fraction. In order to get the distribution of halos that make $M$ at $\delta_0$, one can proceed (approximately) as follows:

1) Choose $m_1$ with probability $f(m_1, \delta_1 / M, \delta_0)$

2) we are left with mass $M' = M - m_1$. If $M$ corresponds to $3 m_1$, it means it occupies a volume $V$ such that

$$\frac{M}{V} = f(1 + \delta_0)$$

$\bar{\rho}$ = mean density at time $t_0$.\n
for $m_2$ we have similarly
\[
\frac{m_1}{\nu} = \tilde{g}(1+\delta_i)
\]

And the remaining mass corresponds to a density $\delta'$:
\[
\frac{M-M_1}{\nu-\nu_0} = \tilde{g}(1+\delta')
\]

iii) We can now think of a halo of mass $M'$ at time $\delta'$, and use the same formula, choose $m_2$ with probability
\[
f(m_2, \delta_i / M', \delta')
\]
This will be exact for white noise $P(h) = k^n$ with $n=0$, in which case disconnected volumes are mutually independent.

iv) continue drawing $m_3$, etc until no mass is left, i.e
\[
M = m_1 + m_2 + \cdots + m_N
\]