Stress-Energy Tensor perturbations and SVD decomposition

Last class we discussed the metric perturbations and how they can be decomposed into their scalar ($\xi$), vector ($\nabla\xi$) and tensor ($\Sigma$) components. In order to complete the picture, we need to do the same for the stress tensor and the Einstein tensor, and then write the Einstein equations for perturbations.

Let us consider a fluid (next class we will extend this to a scalar field), for which

$$\mathcal{T}^{\mu\nu} = (\rho + p) \, u^\mu u^\nu + p \, g^{\mu\nu} + \Sigma^{\mu\nu} \quad \begin{cases} \Sigma^\mu_\mu = 0 \\ \Sigma^{\mu}_\nu u^\nu = 0 \end{cases}$$

In order to write down perturbations of $\mathcal{T}^{\mu\nu}$, therefore, we need to consider perturbations of $\rho$, $p$, $u^\mu$, and $\Sigma^{\mu\nu}$, apart from metric perturbations that we already considered. The velocity perturbations is the only non-trivial piece because it involves metric perturbations due to the non-metricity condition $\nabla^\mu u_\mu = -1$.

We have by definition,

$$u^\mu = \frac{dx^\mu}{d\xi} = \frac{dx^\mu}{d\tau} \frac{d\tau}{d\xi}$$

In our coordinate system $x^0 = \tau$, thus we have

$$\frac{dx^\mu}{d\tau} = (1, \frac{dx^i}{d\tau})$$

Now, you may recall from previous course one defines a peculiar...
which is the standard 4-velocity in special relativity, apart from

\[
\frac{d\tau}{dt} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}
\]

We are from here that in the absence of metric perturbation,

\[
\frac{d\tau}{dt} = \frac{d\tau}{dx^0} = a_0 \sqrt{1 + 2\gamma_1 + 2\gamma_2 - (a + 2\gamma_1) \sqrt{1 + 2\gamma_2 - 2\gamma_1(v^2 - \beta^2)}}
\]

which due to the form of our metric,

\[
\frac{d\tau}{dt} = \sqrt{1 - \frac{v^2}{c^2}}
\]

with \(\gamma\) the "usual" 3-velocity. Now we also have

\[
\frac{d\tau}{dt} = \sqrt{1 - \frac{v^2}{c^2}}
\]

where we introduced the conformal expansion scale \(H = \frac{\partial x^i}{\partial \tau}\) physical velocity:

\[
\frac{d\tau}{dt} = \frac{d\tau}{dx^i} = a \gamma^i a_i = a \gamma^i \frac{\partial x^i}{\partial \tau}
\]

Highly non-linear

perturbation by looking at derivations

\[
\frac{d\tau}{dx^i} = \frac{d\tau}{dx^i} = a \gamma^i a_i
\]
the fact that we are using conformal time to take care of the expansion of the universe. In the presence of metric perturbations, the components get changed as the measure of proper distances and times get affected by perturbations.

Now, we are going to work in the limit of small metric perturbations and we will also assume velocities are much smaller than the speed of light, \( v \ll c \), thus we can linearize \( \dot{\alpha} \) and \( \ddot{\alpha} \) in \( \nu \) and metric perturbations. Linearizing, we have

\[
\nu^\Lambda = \frac{1}{\alpha} (1, \nu^i)(1-\alpha) = \frac{1}{\alpha} (1-\alpha, \nu^i)
\]

We will also need the components

\[
\nu_0 = g_{00} v^0 + g_{0i} v^i = \sqrt{-\alpha (1+\alpha)} \quad \nu^i = g_{i0} v^0 + g_{ij} v^j = -\alpha \frac{\nu^i}{\alpha} - \alpha^2 B_i \frac{\nu^i}{\alpha}
\]

Now we are ready to write down the full stress-energy with the only assumption of \( v \ll c \) and small metric perturbations, we have

\[
T^0_0 = -f
\]

\[
T^i_0 = -(g + \nu^i) \nu^0
\]

\[
T^0_i = (g + \nu^0) (\nu^i - B_i)
\]

\[
T^i_j = p \delta^i_j + \Sigma^i_j
\]

and we have used mixed components for simplicity (thus avoiding extra factors due to metric perturbations). Note that we have not
Yet decomposed $f$ and $p$ into background + perturbations - both $\nu$ and $\Sigma$ are perturbations already. The only assumption are $\nu/c^2$ and small metric perturbations (also $\nu, \Sigma$ times metric perturbations are ignored). Using these approximations we can write down conservation of stress energy \(\tau_{ij} = \rho \nabla^2 \phi\). \\
\[
\begin{align*}
\frac{\partial f}{\partial t} + 3(H+\dot{\Omega})(f+p) + \nabla \cdot \left[(f+p) \partial^2 f\right] &= 0 \\
\frac{\partial}{\partial t} \left[(f+p)(\nabla^2 f)\right] + 4H(f+p)(\nabla^2 f) &= -\nabla p - \nabla \cdot \Sigma - (f+p) \nabla \Phi
\end{align*}
\]
Note that $f$ is energy density law, and $p/\rho$ is $(f+p)$ appears because the change in energy density is also affected by $\nabla^2 \dot{\phi}$ work due to expansion for a relativistic fluid in which $p$ is comparable to $f$. Also note in expansion term, $\nabla^2 f$ appear, that's because the perturbed scale factor is $a(t)$. In the spatial part, thus the effective conformal Hubble is $\nabla^2 f$ has no trace. 

The appearance of $\nabla^2 f$ in the momentum conservation equation is due to the fact this is the velocity in a frame where the shift vector is zero, in other words, $\nabla^2 f$ takes care of the fact that $\dot{\Sigma}$ is due to the shift vector. The rest of the terms are standard from Newtonian conservation, note $\nabla \Phi$ is the gradient of the gravitational potential.

The standard procedure would be now to linearize $\tau_{ij}$, i.e. assume density and pressure have small perturbations compared to their background value, and similarly for $\Sigma$ and $\nabla^2 f$. One then does the SIFT decomposition for $\tau_{ij}$ which is equivalent to label $\Sigma$ decompose $\nabla^2 f$ into $S$ and $V$, and $\Sigma$ into $S, V$ and $T$ as we did last class. Before we say more about this, let us consider again gauge transformation.
Gauge Transformations (again)

Let's consider how perturbations transform under a gauge transformation

\[ X^\mu = x^\mu + \xi^\mu(x) \]

Consider the metric again. We want \( \bar{g}_{\mu
u}(x) \):

\[ \bar{g}_{\mu\nu}(x) = \tilde{g}_{\mu\nu}(x) - \tilde{g}_{\mu\nu}(x) = \tilde{g}_{\mu\nu}(x) - \tilde{g}_{\mu\nu}(x) 3^\alpha + O(3^2) \]

where \( \tilde{a} = \frac{\partial}{\partial x^\lambda} \), which we can set to \( \frac{\partial}{\partial x^\lambda} \) working to first order in PT.

But:

\[ \tilde{g}_{\mu\nu}(x) = \frac{\partial x^\alpha}{\partial x^\mu} \frac{\partial x^\beta}{\partial x^\nu} \tilde{g}_{\alpha\beta}(x) = (\delta^\alpha_\mu - \tilde{e}^\alpha_\mu)(\delta^\beta_\nu - \tilde{e}^\beta_\nu) \tilde{g}_{\alpha\beta}(x) \]

\[ \Rightarrow \bar{g}_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) - \tilde{e}^\alpha_\mu \tilde{g}_{\alpha\nu} - \tilde{e}^\beta_\nu \tilde{g}_{\mu\alpha} - \tilde{g}_{\mu\nu}(x) 3^\alpha \]

to first order in PT.

Now, we decompose our gauge transformation into SVT:

\[
\begin{cases}
\xi^0 = \alpha(x) & (1S) \\
\xi^i = \nabla_i \beta + \epsilon_i & \text{with } \nabla_i \epsilon = 0, & (1S + 2V)
\end{cases}
\]

Then we have

\[
\begin{align*}
\tilde{B}_i &= B_i - \nabla_i (\alpha - \beta) - \epsilon_i \\
\tilde{D} &= D - \frac{1}{3} \nabla^2 \beta - \nabla \alpha \\
\tilde{E}_{ij} &= E_{ij} - \nabla_i \beta - \frac{1}{2} (\nabla_i \epsilon_i + \nabla_j \epsilon_j) \\
\end{align*}
\]

\[ \tilde{E}_{ij}^s = E_{ij}^s - \frac{1}{2} \left( \nabla_i \epsilon_j + \nabla_j \epsilon_i \right) \]

\[ \tilde{E}_{ij}^v = E_{ij}^v - \frac{1}{2} \left( \nabla_i \epsilon_j + \nabla_j \epsilon_i \right) \]

and \( \tilde{E}_{ij}^{SVT} = E_{ij}^{SVT} \) (gauge invariant)
One can similarly do this for $T_{\mu\nu}$ rather than $G_{\mu\nu}$, and get the transformation laws for $x, p, v, \xi, \eta, \delta$, etc. Let us assume $p=0=\xi$ for simplicity. Then:

$$\tilde{T}^{\mu}_{\nu} = \theta^{\mu}_{\nu} + \gamma^{\mu}_{\nu} \theta, \lambda + \delta^{\mu}_{\nu}(\gamma^{\lambda})$$

$$\Rightarrow \delta = \delta + 3\gamma + \alpha$$

and for the velocity,

$$\tilde{v}^{\mu} = v^{\mu} + \gamma^{\mu}_{\nu}, \nu \Rightarrow \begin{cases} \tilde{v}^i = v^i + \gamma^i_{\nu}, \nu = \nu^i_{\nu} + \beta_{,\nu} \\ \tilde{v}^i = v^i + \nu^i_{\nu} + \epsilon_{,\nu} \end{cases}$$

Let us now consider some gauges. First notice we have $2\delta$ and $2\nu$ dof at our disposal to play with. In the synchronous gauge, one sets

$$\begin{cases} g_{00} = -1 & \Rightarrow A = 0 \\ g_{0\nu} = 0 & \Rightarrow A = 0, B = 0 \end{cases}$$

then $\nu^0 = 0$, $B^0 = B^i = 0$

The synchronous gauge has the property that there is a set of freely falling observers that have constant spatial coordinates, that is, $\nu^i = 0$. This follows from the geodesic equation

$$\frac{dv^\mu}{dt} + \Gamma^\mu_{\nu\rho} v^\nu v^\rho = 0$$

for $dx = \sqrt{-g} dt$ and $v^\mu = \frac{dx^\mu}{dt}$. When $A = 0$, $B^i = 0$, it follows that

$T^0_{\mu 0} = 0 \Rightarrow \nu^i = 0$ is a geodesic. In other words, each observer carries a clock reading conformal time $\tau$ and a fixed spatial coordinate label $\xi^i$. Obviously the slicing is perpendicular to the geodesics, so the shift is zero ($B^i = 0$).
The synchronous gauge has some peculiarities. First, it is not just a gauge but a family of gauges: one can do coordinate transformations with \( \beta = \beta_0(z) \frac{dz}{a(z)} \) and \( E = E(z) \) and still satisfy the gauge conditions. This arises from the freedom to adjust the initial setting of clocks and coordinate labels of the observers following geodetics. Also, because it is a set of Lagrangian coordinates (with \( xi \) fixed to observers), when the trajectories cross coordinates become singular. This only happens when density fluctuations are "large", i.e., \( \delta \rightarrow 1 \), so it is not a concern when dealing with early universe.

The synchronous gauge is good for numerical solutions because of its stability (e.g., constant vorticity), though the metric perturbations are a bit hard to interpret since there is no analog of Newtonian potential. One can, e.g., solve on this gauge and then transform to another gauge for easier interpretation if desired.

The Poisson gauge corresponds to imposing the gauge conditions,

\[
\begin{align*}
\nabla \cdot B &= 0 \\
\n\nabla \cdot E &= 0
\end{align*}
\]

Thus in this case one eliminates the scalar part of \( \Phi \), and the scalar and vector parts of \( E_{ij} \), again, for a total of 25 and 2V conditions. Thus, the only degrees of freedom left are

\[
\begin{align*}
A, \phi & \quad (23) \\
B_i^\nu & \quad (2V) \\
E_{ij}^T & \quad (2T)
\end{align*}
\]

For the linear approximation \( \delta, \nu \) and \( \rho \) perturbation evolve independently, and the scalar part of the metric \( \Phi \) thus
The Einstein equations in the Poisson gauge are

\[ G^0_0 : \nabla^2 D + 3H (D + HA) = -4\pi G a^2 \delta \]

\[ (G^0_i)_S : \nabla_i (D - HA) = 4\pi G a^2 \left[ (\varphi + \rho) (\varphi - \beta_i) \right]_S \]

\[ (G^0_i)_V : \nabla^2 \beta_i^V = -16\pi G a^2 \left[ (\varphi + \rho) (\varphi - \beta_i) \right]^V \]

\[ G^i_i : -\nabla_i^2 \left( A - 2\delta \right) + (2\nabla^2 \beta_i) A + \frac{1}{3} \nabla^2 (\nabla^2 \beta_i) = 4\pi G a^2 (p - \rho) \]

\[ (G^i_{j+k})_S : \xi_{ij} (D + A) = -8\pi G a^2 \Sigma^S_{ij} \]

\[ (G^i_j)_V : - (\partial_i + 2H) \nabla_i \beta_j^V = 8\pi G a^2 \Sigma^V_{ij} \]

\[ (G^i_{j+k})_T : (\partial_i^2 + 2H \partial_i - \nabla^2) \Sigma^T_{ij} = 8\pi G a^2 \Sigma^T_{ij} \]

Of course, not all of these equations are independent, they are related by energy-momentum conservation. One can take either the first 3 or the second 3 (or some linear combination) plus the last one (for tensor modes).

What is interesting about the Poisson gauge is that one can obtain the scalar and vector potentials directly from the instantaneous stress-energy distribution with no time integration required, i.e., matter and metric perturbations are algebraically related (except for the tensor modes). We can take three equations to be

\[ \nabla^2 \beta_i^V = -16\pi G a^2 \left[ (\varphi + \rho) (\varphi - \beta_i) \right]^V \]

\[ \xi_{ij} (D + A) = -8\pi G a^2 \Sigma^S_{ij} \]

and combining the first two above we have

\[ \nabla^2 D = -4\pi G a^2 \left[ \frac{3}{5} \delta + 3\nabla^2 \right] \quad \text{with} \quad -\nabla^2 \delta \equiv (\varphi + \rho) (\varphi - \beta_i) \right]_S \]
We can now take a look at the Newtonian limit from here. First, if there is no expansion, $H = 0$ and the equation for $D$ can be written as Poisson equation

$$\nabla^2 A = 4\pi G a^2 \delta$$

When there is expansion, there is an additional source to energy density, that is $\delta$ which is the scalar part of momentum density. We can estimate the relativistic correction by

$$\nabla^2 \delta \approx \frac{\nabla^2 \delta}{k}$$

$$\Rightarrow \frac{3\nabla^2 \delta}{\delta^2} \approx 3 \frac{\nabla^2 \delta}{k} \approx 3 \left( \frac{kH}{\delta} \right)^2 \approx 3$$

where we have defined the horizon wave-vector $k_H = \frac{2\pi}{cH(a)} \approx \frac{2\pi}{cH(\text{H})} \approx \frac{1}{3000} \text{Mpc}$

Thus the correction only becomes important at distances comparable to the horizon. Similarly, stresses are typically small, then $\Sigma \approx \left( \frac{c}{\delta} \right)^2$, with $c$ the characteristic thermal speed of particles, thus $\Sigma \propto D$ to order $(\Sigma c)^2$. Also, $B \nabla$ is negligible due to no vorticity.

The physical interpretation in this gauge is simple since we have the Newtonian potential + small corrections, and Do-A gives the perturbative (small) corrections to the local scale factor. One inconvenience of this gauge is that when relativistic corrections become important at horizon scale, the extra term in the Poisson equation leads to numerical instabilities.