Classification of Inflationary Models

In order to constrain inflationary models against observations, it is useful to classify them according to where they land in parameter space of observables, such as \((n_s - 1)\) and \(r\) (to lowest order in slow-roll). Even in the context of single-field inflationary models, the number of models proposed is very large; however, they can be generally characterized by two mass scales in the potential: a height \(A^4\) (corresponding to the energy density during inflation) and a width \(\mu\) (corresponding to the change in the field value \(\Delta \phi\) during inflation), that is

\[ V(\phi) = A^4 f(\frac{\phi}{\mu}) \]

The height \(A\) is fixed by the normalization of the density perturbation for a given model (through \(V(0)\)); thus, basically \(\mu\) is the only relevant parameter left. Different models have different \(f's\). The relevant parameter space for distinguishing models to lowest order is the \(r - n_s\) plane. Since

\[ n_s - 1 = 2 \eta - 6 \epsilon \]
\[ r = 16 \epsilon \]

The relationship between \(n_s\) and \(r\) is through \(\eta\); models can be classified in the \(r - n_s\) plane through the value of \(\eta\).

1) Small-field models: \(\eta < 0.4\)

These are the type of potentials that arise naturally from spontaneous symmetry breaking (e.g., new inflation and natural inflation) \(\to\) cos-type potential \(\Phi\) that starts near an unstable equilibrium (defined as the origin) and rolls down the potential to a stable minimum.
Typically $\epsilon$ is quite small in this case (e.g. for the potential given above, we have

$$\epsilon = \frac{p^2}{2} \left( \frac{M_\chi^2}{\mu} \right)^{2p-2}$$

whereas: $$\eta = \rho \left( p-1 \right) \left( \frac{M_\chi^2}{\mu} \right)^{p-2}$$

Remember $\phi \ll \mu$, so for $p > 2$, $\epsilon$ is small. To relate ($\phi - 1$) and $r$ we introduce again the number of e-folds $N$, which remember is a parametrization for the time when a given mode that we see crosses the Hubble, or also you can think of it as the value of the field $\phi$ at that crossing Hubble time.

$$N = \int_{\phi}^{\phi_0} \frac{H \, dt}{\dot{H}^2} = \int \frac{H \, d\phi}{\dot{H}^2} = \frac{1}{\frac{M_\chi^2}{\mu}} \int \frac{V}{V_1} \, d\phi$$

$$\frac{3H^2}{\dot{H}^2} = V = \frac{\rho}{3M_\chi^2} = H^2$$

$$\frac{\rho}{V_1} = \frac{1}{\frac{M_\chi^2}{\mu}}$$

$$\epsilon = \frac{1}{2} \left( \frac{\rho}{V_1} \right)^2$$

$$\frac{p^2}{p(p-1)} \left( \frac{M_\chi^2}{\mu} \right)^{2p-2}$$

$$\Rightarrow N \approx \frac{\rho}{p(p-1)} \left( \frac{M_\chi^2}{\mu} \right)^{2p-2} \left( \frac{\dot{\phi}}{\dot{\phi}_0} \right)^2$$

Then we have

$$\eta = \frac{\rho}{p(p-1)} \left( \frac{M_\chi^2}{\mu} \right)^{2p-2} \left( \frac{\dot{\phi}}{\dot{\phi}_0} \right)^2$$

$$\Rightarrow \eta = -2p(p-1) \left( \frac{M_\chi^2}{\mu} \right)^{2p-2} \left( \frac{\dot{\phi}}{\dot{\phi}_0} \right)^2 \approx -2p(p-1) \left( \frac{\dot{\phi}}{\dot{\phi}_0} \right)^2$$

$$\Rightarrow \eta > 0$$

Note in this case we didn't need to evaluate $\dot{\phi}_0$, because of the form of the potential and small-field condition, but otherwise one has to include that (integral for $N$ is typically dominated by one limit) $\dot{\phi}_0$ results from $\epsilon = 1$ or $\eta = 1$, whichever...
happens first. Note that for the particular case $p=2$ it follows that
\[ r = 5 \left( 1 - r_5 \right) \in [1 + N(1 - r_5)] \]

\[ \text{ii) Linear models ($\eta = 0$)} \]

In this case $V(\phi) \propto \phi^4$, then $r = \frac{5}{3} (1 - r_5)$

\[ \text{iii) Large-field models ($0 < \eta \leq 2\pi$)} \]

This is typical of "chaotic" inflation scenarios where the scalar field is displaced from the minimum of the potential by an amount of a few $\phi_p$ - typical potential is
\[ V(\phi) \sim N \left( \frac{\phi}{\phi_p} \right)^4 \]

or exponential potentials $V(\phi) \sim N \phi^4 e^{4\phi} \, d\phi$, for which it follows that $r = \frac{5}{3} (1 - r_5)$. For $V \propto \bar{\phi}^4$, we have
\[ r = 5 \left( \frac{p+1}{p+2} \right) (1 - r_5) \]

As we mentioned last time, this line (for fixed $p$) is parameterized by $N$, e.g. for $p=4$:
\[
\begin{align*}
N &> \frac{10}{N \phi_p} \\
1 - r_5 &> \frac{3}{N \phi_p} \quad \text{(Again, talk is negative)}
\end{align*}
\]

\[ \text{iv) Hybrid models ($\eta = 2\pi$)} \quad \left( \text{i.e. } \frac{V''}{V} > \left( \frac{V'}{V} \right)^2 \right) \]

These type of models appear extremely frequently when trying to realize inflation in supersymmetric models. The inflaton evolves towards a minimum for large $\phi$ to small $\phi$ when an instability happens at $\phi = \phi_c$ that terminates inflation; this is due to a second field that develops a negative effective square mass, e.g.

\[ \begin{align*}
V &\propto N \phi^4 e^{4\phi} \\
\phi &\to \phi_c \\
\text{inflation end} &\to \text{end}
\end{align*} \]
A generic potential of this type is something like
\[ V = V_0 \left[ 1 + \left( \frac{\phi}{\mu} \right)^2 \right] \]

Since the end of inflation is determined by other physics, there is a second parameter characterizing the models. Because of this extra freedom, hybrid models fill a broad region in the $\eta$-$r$ plane, though there is no overlap with previous models. When $\eta > 3\epsilon$, we can have a positive tilt. This is a distinct feature of this model, though it is pole. One can also have a negative tilt ($2\epsilon < \eta < 3\epsilon$).

Summarizing:

![Graph showing regions of linear, small field, large field, and hybrid models.]

Evolution of "Superhorizon" Perturbations

Before we can make contact with observations, we have to consider what happens after perturbations cross the Hubble radius and become "superhorizon", and then cross back in during radiation and matter domination epochs. Since we discuss this already in detail in Chapter 3, I will be brief.

The important point is that the surrounding curvature $\epsilon$ (which we called $R$ in cosmos) is conserved at scales larger than $H^{-1}$ for advected perturbations (which in flatness generation). We work in large-scale limit, $k \ll \frac{1}{H}$.

For $\frac{b}{a} \ll 1$, the conservation of $\delta T^0_0$ gives
\[ \frac{3}{2e} \delta T^0_0 + 3H \delta T^0_0 - \Lambda \delta T^0_0 = -3 \left( \rho + p \right) \frac{\delta \phi}{\delta t} \]
with the gravitational potential $\Phi$ related to $\delta$ by $(k/4\pi)^2$:

$$\Sigma = -\Phi - \frac{1}{3} \frac{\delta T}{\rho_p}$$

We use these two to find an equation for $\delta$:

$$\frac{\delta T^0}{\delta t} + 3H \delta T^0 - H \delta T^i = 3(\rho_p) \frac{\delta p}{\delta t} + (\rho_p) \frac{\rho}{\delta t} \left[ \frac{\delta T^0}{\rho_p} \right]$$

Note the $\delta T^0$ cancel, so we have

$$\delta T^0 \left[ 3H + \frac{1}{\rho_p} \left( \frac{dp}{dt} + \frac{dp}{dt} \right) \right] - H \delta T^i = 3(\rho_p) \frac{\delta p}{\delta t}$$

Since $\frac{dp}{dt} = -3(\rho_p) H \delta p$ ($g \propto a^{-3(1+w)}$), the first two terms cancel and we are left with

$$\frac{\partial \delta}{\partial t} = -\frac{1}{3(\rho_p)^2} \left[ H(\rho_p) \delta T^i - \delta T^0 \frac{\partial \rho}{\partial t} \right] + \frac{1}{3} \frac{dp}{dt} 3dp - \frac{dp}{dt} = 0$$

for inflationary perturbations

$$\delta T^0 = \frac{2 + 3w}{5 + 3w} \delta$$

assuming $w$ does not depend on time. Thus $\delta$ is also conserved at scales larger than Hubble. During $\Lambda$CDM transition $w$ changes and there is a factor of $\delta/10$ decrease in $\delta$, but apart from this we can think of $\delta$ as being constant at large scales.

We can use this fact to give an argument why the 3-point function of $\delta$ should be small (see Maldacena paper for details). Consider

$$\langle \delta(k_1) \delta(k_2) \delta(k_3) \rangle = B^2 \frac{\delta}{12\pi} \delta_D(k_1+k_2+k_3)$$

we know

$$\langle \delta(k) \delta(k) \rangle = P(k) \delta_D(k_{\text{THK}})$$

with $P(k)/A = 2\pi^2 H^2 \frac{1}{k^3} \epsilon$

Consider $k_3 < k_1, k_2$, i.e. a triangle like

$$k_3 \quad k_1 \quad k_2$$
Since $k_3$ is so much smaller than $k_1$ and $k_2$, it crosses the Hubble radius during inflation much earlier than $k_1$ and $k_2$. By the time $k_1$ and $k_2$ cross $H^{-1}$, $\xi_3$ is then constant, and the only effect this can have on $\xi_2, \xi_1$ is through the fact that a change non-zero $\xi$ is a slight change in the density of the universe (since $\xi$ is curvature, or local scale factor). Then for a non-zero $\xi_3$, $k_1$ and $k_2$ will cross $H^{-1}$ slightly earlier by $\delta t$ given by $-\frac{\xi_3}{\xi_1} \delta t = \delta T_k$. Then

$$\langle \xi_1 \xi_2 \xi_3 \rangle \sim -\langle \frac{\xi_3}{\xi_1} \frac{\xi_2}{\xi_1} \rangle \frac{1}{\xi_1^2} \frac{d \xi_1}{d \xi_2} \langle \xi_1 \xi_2 \rangle$$

but the only dependence on $\delta T_k$ comes from the dependence of $H^2/\xi_3$, which gives rise to a tilt factor, then:

$$B_{1,2,3}^0 \sim (\delta t)^{-1} P_\xi^2$$

Since $(\delta t)^{-1} \lesssim 0.05$, this is a non-gravitational of only a few percent in the primordial potential $\Phi$.

Similarly, as we saw in cosmic $\delta_k$, from the evolution of $\Phi$ we can derive the shape of the transfer function, i.e., the spectrum after all modes have entered the Hubble radius.

Recall that once modes enter in $H^{-1}$ era, $\Phi$ decays, since $\delta$ only grows logarithmically. Then from Poisson equation:

$$\nabla^2 \delta \sim H^2 a^2 \delta \sim a^{-2} \delta \sim a^{-2}$$

$\Phi$ decays as $a^{-4}$

For modes that enter in the MMT era, $\delta \propto (\text{assume } \delta \propto k_4 \text{ for simplicity})$

$$\nabla^2 \Phi \sim H^2 a^2 \delta \sim H^2 a^3 \sim \text{const}.$$  

Thus there is an $a^4$ suppression between high-$k$ and low-$k$ modes. This
translate into a $k^4$ suppression of power, since suppression by $(\frac{a}{a_e})^4$ (where $a_e$ is at NH-RAD equality, and $a$ is when mode entered in RAD) is equal to $(\frac{a}{a_e})^4$ at NH-RAD era. Then we have for today’s power spectrum of density perturbations

\[ P(k) \propto h^{-3} \]  
\[ (\log P(k)) \rightarrow \log h \]

$h_e$ *is essentially the size of $H^{-1}$ at NH-RAD equality.*