Beyond Standard Cosmology

We mentioned that as a decrease in the past, we go from our current dark-energy dominated universe to matter dominated and then radiation dominated era. But this picture is not sufficient to explain some aspects of the observed universe, as we now discuss.

To begin, let's recall the concept of the Hubble radius:

\[ R_{	ext{H}}(t) = \frac{c}{H(t)} \]

where \( H(t) \) gives the characteristic time-scale at time \( t \), which one can also think of as the surface at which the recession velocity equals the speed of light:

\[ c = v = \frac{dr}{dt} \implies t = \frac{c}{H} \]

The Hubble radius is important as it is a local (in time) measure of the characteristic time scale, and it shows up in equations of motion (this is as opposed to the horizon, which is an integral over the past light cone).

Another important concept is the physical distance of a worldline of an object that is at comoving coordinate \( \lambda_{\text{comov}} = \text{const.} \) away from us. This is also referred to as a "wavelength" because we can think of a given comoving scale \( \lambda_{\text{comov}} = \text{const.} \) as being stretched by the expansion of the universe:

\[ \lambda_{\text{phys}} = a(t) \lambda_{\text{comov}} = a(t) \lambda_{\text{comov}} \]

like any physical distance.

Now we note that since \( H = \frac{\dot{a}}{a} \) and \( \int a^{-2/3} \text{ d}t = \text{const.} \),
we have for the Hubble radius

\[
\begin{aligned}
H^{-1} &= \begin{cases} 
\left(\frac{2}{3t}\right)^{-1} & \text{MAT} \atop 
\frac{2}{2t} & \text{RAD}
\end{cases} \\
& \sim t \sim a^{3/2} & \text{MAT} \\
& \sim t \sim a^2 & \text{RAD}
\end{aligned}
\]

whereas \( \lambda_{\text{phys}} \sim a \) always. Let's plot these in a log-log plot:

![Log-log plot with different lines showing evolution of physical scales and Hubble parameter over time](image)

as we won't longer more wavelengths become "causally connected," i.e. increasing number of \( \lambda_{\text{phys}} \) become smaller than the Hubble radius \( H^{-1} \) which is \( \sim c \times \text{time scale} \); we say that a wavelength "crosses the Hubble radius" when \( \lambda_{\text{phys}} = H^{-1} \)

Note that if \( a \sim t^n \) with \( n \leq 1 \) then we have always

\[ \lambda \sim a \sim a t^n \quad \text{but} \quad H^{-1} \quad \text{grows slower than} \ H^{-1} \]

\[ \Rightarrow \text{more and more scales become in causal contact, which is a familiar behavior. However, it introduces a puzzle in the standard big bang model known as the horizon problem:} \]

Why is the Universe so smooth on large scales? More precisely, why is the temperature of the cosmic microwave background (CMB)
In terms of the same temperature, how did all these causally connected regions happen? This is the horizon problem.

To agree on the same temperature, all H-decay, and make Tcmb the same as H, the size of the causally connected region corresponds today to a volume of H-decay at that time, the last time that photons interacted (when the universe became deuterium enough), 2x10^-3.

The sky we see today is not the same as H and Tcmb is pretty uniform, inside this volume, the CMB was released from interactions of side H and Tcmb is pretty uniform. However, H-decay of side H, and Tcmb is pretty decoupling (when the universe became deuterium enough).
Another puzzle in the standard big bang model is the so-called **flatness problem** - From the Friedmann equation we have

\[ \Omega = \frac{1}{3H^2} \Rightarrow \Omega - 1 = \frac{k}{a^2H^2} = \frac{k}{a^2} \approx \int t \text{RAD} \left( a t^{1/2} \right) \text{MAT} \left( a t^{3/2} \right) \]

if \( \Omega = 1 \Rightarrow k = 0 \) and this is always so (no true dependence)

However, if \( \Omega + 1 \Rightarrow |\Omega - 1| \) increases with time. For example, during RD domination we have

\[ |\Omega - 1| \ll 1 \text{ where today we know } |\Omega - 1| \ll \mathcal{O}(1) \]

then if we extrapolate back (assuming RD and MAT for simplicity) to the time the light elements were created in big bang nucleosynthesis (BBN), which is the earliest moment where we have direct evidence our extrapolation back in time works, we have

\[ |\Omega - 1| \ll \text{BBN} \ll 100 \text{ sec} \approx 10^{-16} \]

So \( |\Omega - 1| \) would have been to be fine-tuned to a part in \( 10^{16} \) at BBN! Otherwise today we would have \( \Omega \gg 1 \) or \( \Omega < 1 \):

\[ \begin{align*}
|\Omega - 1| & \ll 10^{-16} \\
\text{For } & \text{ today we have } |\Omega - 1| \ll 10^{-16} \\
\text{(again, fine-tuning)} &
\end{align*} \]

**Inflation**

The horizon problem arises because during the evolution of the scale factor in RD, MAT \( H^{-1} \) always scale faster than \( \Delta a \) and

\[ \frac{\Delta a}{a} \ll 10^{-16} \]
If there is a long enough epoch in the past where that trend is reversed, we can have $\dot{a} < 0$ early enough and allow for causal contact back then.

so, in std cosmology $\frac{d}{dt} \left( \frac{H^{-1}}{a} \right) = \frac{d}{dt} \left( \frac{H^{-1}}{a} \right) > 0$

to reverse this we need $\frac{d}{dt} \left( \frac{H^{-1}}{a} \right) = -\frac{\ddot{a}}{a^2} < 0 \Rightarrow \ddot{a} > 0 \text{ (Inflation)}$

that is, an epoch of acceleration long enough in the early universe can solve the "horizon problem". In picture,

for simplicity to make this graph we assume an acceleration epoch $\ddot{a} > 0$ with $a \propto e^{H_{inf}t}$ with $H_{inf} = \text{const.}$ so $H^{-1}$ is a horizontal line during inflation. We see then that the $H_{0}^{-1}$ scale today was altogether in causal contact back during inflation if we require $a_{begin}$ to be early enough.

Such a period of accelerated expansion during the early universe also solves the flatness problem since

$$\Omega - 1 = \frac{k}{(cH)^2} v \frac{k}{ e^{-2H_{inf}t} \rightarrow 0 \text{ exponentially}}$$
so, if inflation goes on over many e-folding times, $\Omega \to 1$ by the end of inflation with exponentially small deviations. Typically to require $H_{\text{rel}}$ to be in causal contact gives $\sim 60$ e-foldings of inflation so