Tidal Force in Newtonian Gravity

To see curvature one needs to study the relative motion of at least two particles, as we already discussed before. The reason is that for a single object one can always choose a frame that is freely falling with the object and thus it will remain at rest in that frame. Studying relative motion of two or more objects we can see that gravity is present, i.e., consider two freely falling labs in outer space and in a gravitational field:

\[ \frac{d^2 x^i}{dt^2} = -\frac{\partial \Phi(x)}{\partial x^i} = -8\hat{u} \frac{\partial \Phi(x)}{\partial x^j} \]

For each particle we have

\[
\frac{d^2(x^i+x^i)}{dt^2} = -8\hat{u} \frac{\partial \Phi(x^i+x^i)}{\partial x^i} \]

Taylor

\[
\Rightarrow \left[ \frac{\partial \Phi(x^i+x^i)}{\partial x^i} + \frac{\partial^2 \Phi(x^i+x^i)}{\partial x^i \partial x^j} \right] \hat{u}^2 x^k \]

\[
\Rightarrow \frac{d^2 x^i}{dt^2} = -8\hat{u} \left( \frac{\partial \Phi}{\partial x^i} \right) x^k
\]

Newman's deviation eqn. for separation vector \( \hat{x} \).
The tensor \( \frac{\partial^2 \Phi}{\partial x^i \partial x^j} \) is the tidal gravitational acceleration tensor.

Note that the field eqn. in Newtonian gravity, the Poisson eqn.

\[ \nabla^2 \Phi = 4\pi G \mu \]

\( \mu \) mass density

can be written in terms of the tidal tensor as

\[ 8\pi \frac{\partial^2 \Phi}{\partial x^i \partial x^j} = 4\pi G \mu \]

(a similar relationship) (will hold in GR!)

**Geodesic Deviation Equation**

We are now ready to generalize the previous discussion to GR, where the analogous object to the tidal tensor will give us a local measure of spacetime curvature.

Now we study the separation 4-vector \( \bar{x} \) between nearly geodesics as a function of proper time. Since the derivative of any scalar function \( f \) along the geodesic is

\[ \frac{df}{d\tau} = \frac{\partial f}{\partial x^i} \frac{dx^i}{d\tau} = u^i \frac{df}{dx^i} = \nabla_{\bar{u}} f \]

i.e. the covariant derivative along \( \bar{u} \), for the 4-vector \( \bar{x} \) we have similarly

\[ \bar{v} = \nabla_{\bar{u}} \bar{x} \]

\[ \bar{w} = \nabla_{\bar{u}} \nabla_{\bar{u}} \bar{x} \]

for the velocity and acceleration of the separation vector. Now we use the expression for cov. der.

\[ v^a = u^\beta \partial_\beta x^a = \left( u^\beta \frac{\partial x^a}{\partial x^\beta} \right) + T^a_{\beta \gamma} u^\beta \gamma \]

\[ w^a = u^\sigma \partial_\sigma x^a = \frac{dv^a}{d\tau} + T^a_{\sigma \delta} u^\sigma \gamma \]

\( \gamma \) torsion
Clearly, $v^\alpha$ is linear in $x^\alpha$ and $w^\alpha$ is linear in $x^\alpha$, and $w^\alpha$ will be linear in $x^\alpha$, after some tedious algebra one gets,

\[
(\nabla_v \nabla_w x)^\alpha = - R^\alpha_{\beta \gamma \delta} \, u^\beta \, v^\gamma \, w^\delta
\]

for the relative acceleration between nearby geodesics. This is the Geodesic Deviation equation, where the object

\[
R_{\beta \gamma \delta}^\alpha = \frac{\partial v^\alpha}{\partial x^\beta} - \frac{\partial v^\alpha}{\partial x^\delta} + \Gamma^\alpha_\beta \, v^\gamma \, \Gamma^\gamma_{\delta \epsilon} - \Gamma^\alpha_\delta \, v^\gamma \, \Gamma^\gamma_{\beta \epsilon}
\]

is the Riemann curvature tensor. [note it maps vectors $u, v$ into acc. vector $\vec{\alpha}$] - We will explore its properties shortly (see homework!)

Let's see first that this makes sense. Evaluate this eqn. in the closest possible to an inertial frame, i.e. the freely falling frame where

\[
\nabla_u \vec{\alpha} = 0
\]

and similarly \( \nabla_w \vec{\alpha} = 0 \)

Now:

\[
(\vec{\alpha}_\hat{\alpha})_\beta \quad (\nabla_u \nabla_w x)^\alpha = \vec{\alpha}_\hat{\alpha} \cdot (\nabla_u \nabla_w x) = \nabla_u \nabla_w (\vec{\alpha}_\hat{\alpha} \cdot x)
\]

\[
\nabla_u \vec{\alpha} = 0
\]

\[
\vec{\alpha}_\hat{\alpha} \quad \text{is a scalar}
\]

\[
\text{Rem.: } \alpha \text{ & } \hat{\alpha} \text{ are different indices. } \vec{\alpha} \text{ denotes it re-expressed to an orthonormal basis}
\]

RHS has \( \vec{\alpha}_\hat{\alpha} R^\alpha_{\beta \gamma \delta} = R^\alpha_{\beta \gamma \delta} \)

Now \( u^\beta = u^\beta (\vec{\alpha}_\hat{\alpha})^\beta \)

\[
\tilde{u} = u^\beta \vec{\alpha}_\hat{\beta}
\]

\[
(\vec{\alpha}_\hat{\alpha})_\beta \quad \text{is a component in coordinate basis of}
\]

the $\beta$th orthonormal basis vector.
\[ u^\delta = u^\delta (e_\delta^\beta) \] and \[ \chi^\tau = \chi^\tau (e_\delta^\beta) \]

then have

\[ R^\alpha_{\beta \gamma \delta} (e_\beta^\rho) (e_\gamma^\delta) (e_\delta^\tau) \]

\[ u^\beta \chi^\tau u^\delta \]

but in FFF \[ u^\beta = \delta \]

\[ R^\alpha_{\beta \gamma \delta} : \text{ components of Riemann in FFF} \]

\[ \frac{dx^\alpha}{dt^2} = - R^\alpha_{\delta \gamma \delta} \chi^\delta \]

(FFF)

Let's see that this reduces to the Newtonian case in the weak-field static metric

\[ ds^2 = -(1+2\Phi) dt^2 + (1-2\Phi) (dx^2 + dy^2 + dz^2) \]

which is valid in the Newtonian limit.

Since curvature vanishes for \( \Phi = 0 \), \( R \) will be linear in \( \Phi \), so to work in that order we can ignore the difference between coordinate basis vectors (given by coordinates \( t, x, y, z \) above) and orthonormal basis vectors since they will be different only to order \( \Phi \). So we can just obtain for spatial case

\[ \frac{d^2 x^i}{dt^2} = - R^i_{\ j \ o \ j} \chi^j \]

Compare this with Newtonian case:

\[ \frac{d^2 x^i}{dt^2} = - \frac{\partial^2 \Phi}{\partial x^i \partial x^j} \chi^j \]

Now

\[ R^i_{\ j \ o \ j} = \frac{\partial}{\partial x^o} + \frac{\partial}{\partial x^j} - \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^o} \]

So we can neglect them in \( \Phi \ll 1 \)
\[ \Pi^i_{00} = \frac{1}{2} \left( \frac{\partial g^{ij}}{\partial x^0} + \frac{\partial g^{ij}}{\partial t^0} - \frac{\partial g^{00}}{\partial x^0} \right) = \frac{\partial^2 g}{\partial x_i \partial x^i} \]

\( \gamma^i \) to leading order

\( \Rightarrow \quad R_{00} = \frac{\partial^2 g}{\partial x_i \partial x^i} \quad \checkmark \)

So indeed it reduces in the weak field static limit to Newtonian gravity!

**Properties of Riemann**

To see the symmetry properties it is easiest to evaluate Riemann tensor in a LTF where \( g_{\alpha\beta} = \eta_{\alpha\beta} \) and \( \Pi = 0 \) (but not its derivatives!), i.e., lowering the first index we have (see homework)

\[ R_{\alpha\beta \gamma \delta} = \frac{1}{2} \left[ \frac{\partial^2 g_{\alpha\gamma}}{\partial x^\delta \partial x^\beta} - \frac{\partial^2 g_{\alpha\beta}}{\partial x^\delta \partial x^\gamma} - \frac{\partial^2 g_{\gamma\delta}}{\partial x^\beta \partial x^\alpha} + \frac{\partial^2 g_{\beta\delta}}{\partial x^\alpha \partial x^\gamma} \right] \]

from which it follows

\[ R_{\alpha\beta \gamma \delta} = - R_{\beta\alpha \gamma \delta} = - R_{\alpha\delta \gamma \beta} = R_{\gamma\delta \alpha \beta} \]

antisym.

in first 2

indices

antisym.

in last 2

indices

sym.

under exchange

of two pairs

of indices

and the "cyclic" property

\[ R_{\alpha\beta \gamma \delta} + R_{\beta\gamma \delta \alpha} + R_{\gamma\delta \alpha \beta} = 0 \]

This implies (see homework) that out of the \( 4^4 = 256 \) components of Riemann only 20 are independent. They correspond to the 20 components of 2nd derivatives of the metric that cannot be made to vanish under a general coordinate transformation.
Einstein Equation in Vacuum

As with Newtonian case, the field equation is obtained by tracing over the indices that describe the tidal tensor, i.e., the geodetic deviation. In the GR case it corresponds to introducing a new tensor, the Ricci curvature tensor

$$ R_{\alpha \beta} = R_{\alpha \gamma \delta \beta} \quad \gamma: \text{indices "responsible" of geodetic deviation} $$

which can be written as (in terms of $\nabla^\gamma$'s)

$$ R_{\alpha \beta} = \nabla_\gamma R^\gamma_{\alpha \beta} - \nabla_\gamma \nabla_\delta \Gamma^\gamma_{\alpha \beta} + \Gamma^\gamma_{\alpha \delta} \Gamma^\delta_\gamma_{\beta \rho} - \Gamma^\gamma_{\beta \delta} \Gamma^\delta_\gamma_{\alpha \rho} = R_{\alpha \beta} $$

Analogous to Newtonian case, but in vacuum there are no sources, the RHS of the field equation vanish (as in $\nabla_\gamma \nabla^\gamma = 0$), so the Einstein eqns in vacuum are simply

$$ R_{\alpha \beta} = 0 $$

Now these are 10 non-linear PDE's of second order for 10 metric components $g_{\alpha \beta}$. But, as we discussed, only 6 of the 10 metric components are physical as 4 of them can always be set to zero by appropriate coordinate choice. It turns out, also, that only 6 of the 10 $R_{\alpha \beta} = 0$ equations are independent.
Geometric Interpretation of the Riemann tensor

An important insight into the Riemann curvature tensor comes from examining what happens when we take two covariant derivatives of a vector field $V^\mu$.

$$\nabla_\alpha \nabla_\beta V^\mu = \delta_\alpha \delta_\beta V^\mu + \Gamma^\mu_{\alpha\beta} \nabla_\mu V^\nu - \Gamma^\nu_{\alpha\beta} \nabla_\mu V^\nu$$

to simplify this expression, let's choose a LIF, where $\Gamma$'s vanish (but their derivatives do not) - so we have locally,

$$\nabla_\alpha \nabla_\beta V^\mu = \delta_\alpha \delta_\beta V^\mu + (\partial_\alpha \Gamma^\mu_{\nu\beta}) V^\nu \quad \text{(LIF)}$$

Now consider this expression with $\alpha$ and $\beta$ interchanged,

$$\nabla_\beta \nabla_\alpha V^\mu = \delta_\beta \delta_\alpha V^\mu + (\partial_\beta \Gamma^\mu_{\nu\alpha}) V^\nu \quad \text{(LIF)}$$

Subtracting these, we get the commutator of covariant derivatives,

$$[\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha] V^\mu = (\partial_\alpha \Gamma^\mu_{\nu\beta} - \partial_\beta \Gamma^\mu_{\nu\alpha}) V^\nu \quad \text{(LIF)}$$

where we have used that partial derivatives commute, i.e.

$$[\partial_\alpha \partial_\beta]^\mu_{\nu \lambda} (\partial_\beta \partial_\alpha - \partial_\alpha \partial_\beta) V^\nu = 0$$

What we learn is that in general, covariant derivatives do not commute, i.e. $\nabla_\alpha \nabla_\beta V^\mu$ is not symmetric in $\alpha$ & $\beta$!

Now, since we are in a LIF, we can relate the commutator to the Riemann tensor, so a LIF, so we have

$$[\nabla_\alpha \nabla_\beta V^\mu = R^\mu_{\nu\alpha\beta} V^\nu]$$

Now, although we derived this using a convenient coordinate system (LIF), this is a tensor equation, so it's true in any coordinate (LIF).
System - What this equation means is that in curved spacetime we must be careful to know the order in which covariant derivatives are taken since they don’t commute.

This property is closely related to parallel transporting vectors around closed curves, since such a procedure corresponds to computing first the change in $A^\mu$ in one direction (say $\overline{u}$) then in another (say $\overline{\nu}$), followed by subtracting changes in reverse order (this is what the commutator does). In pictures:

\[ \delta A^\mu = R^\mu_{\nu \lambda \beta} A^\nu \partial_\lambda \partial_\beta A^\mu \]

\[ \delta A^\mu = \left[ \partial_\nu \partial_\beta - \partial_\beta \partial_\nu \right] A^\mu \]

\[ \partial_\nu \partial_\beta A^\mu - \partial_\beta \partial_\nu A^\mu \]

\[ = \left( \nabla_\overline{u} \nabla_\overline{\nu} A^\mu - \nabla_\overline{\nu} \nabla_\overline{u} A^\mu \right) \Delta \overline{u} \Delta \overline{\nu} \]

This lack of commutation of covariant derivatives and its implication for parallel transport along closed curves has a simple illustration for parallel transport on the surface of a sphere:

- After parallel transport, it comes back to the original vector!

This is indeed a measure of the curvature of the sphere. [Give example of cylinder]