**Frobenius' Method**

Now we discuss the method of power series in a more general form. Before we assumed solutions of the form:

\[ y(x) = \sum_{n=0}^{\infty} a_n x^n \]

But there are cases in which solutions may not be regular at \( x = 0 \), e.g.

\[ y = \frac{\cos x}{x^2} = \frac{1}{2} - \frac{1}{2!} + \frac{x^2}{4!} - \cdots \]

or have fractional powers of \( x \) as a factor, e.g.

\[ y = \sqrt{x} \sin x = x^{1/2} - \frac{x^{7/2}}{6} - \cdots \]

To take care of these types of solutions, the Frobenius' method is to generalize the power series expansion to

\[ \left[ y(x) = x^s \sum_{n=0}^{\infty} a_n x^n \right] \]

where \( s \) is a number to be determined from the differential equation; \( s \) can be negative or a fraction.

**Example:** \( x^3 y'' + 2x y' + (x^2 - 2) y = 0 \)

\[ y = \sum_{n=0}^{\infty} a_n x^{n+s} \quad \Rightarrow \quad y' = \sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1} \quad \Rightarrow \quad y'' = \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-2} \]

\[ \Rightarrow \sum_{n=0}^{\infty} \left[ (n+s)(n+s-1) a_n + 2(n+s) a_n - 2a_n \right] x^{n+s} + \sum_{n=0}^{\infty} a_n x^{n+s+2} = 0 \]

\[ \Rightarrow \sum_{n=0}^{\infty} \left[ (n+s)(n+s-1) + 2(n+s) - 2 \right] a_n x^{n+s} + \sum_{n=2}^{\infty} a_{n-2} x^{n+s} = 0 \]
Do usual trick, separate $n=0, 1$ case and then $n=2 \to \infty$:

\[
\left[ 5 (s-1) + 2s - 2 \right] a_0 x^5 + \left[ (5s+2)(s+1) - 2 \right] a_1 x^{s+1} + \sum_{n=2}^{\infty} \left\{ \left[ (n+5)(n+5-1) + 2(n+5)-2 \right] a_n + a_{n-2} \right\} x^n = 0
\]

Then we must have

\[
\begin{align*}
q_0 \left[ 5 (s-1) + 2s - 2 \right] &= 0 \\
q_1 \left[ 5 (s+1) + 2s \right] &= 0 \\
q_n \left[ (n+5)(n+5-1) + 2(n+5)-2 \right] + q_{n-2} &= 0 & n > 2
\end{align*}
\]

Since $q_0 \neq 0$ by assumption, we have:

\[
5 (s-1) + 2s - 2 = 0
\]

which is a quadratic equation, known as the indicial equation, with solutions

\[
s = 1, -2
\]

\[
\Rightarrow q_1 = 0 \quad \text{for both cases by 2nd condition}
\]

These two values will generate the 2 independent solutions of the differential equation. Consider $s = 1$ first,

\[
y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1} \quad q_0 \neq 0, \quad q_1 = 0
\]

The 2nd condition is:

\[
a_n \left[ n(n+1) + 2n \right] = -a_{n-2}
\]

\[
\Rightarrow a_n = -\frac{a_{n-2}}{n^2 + 3n}
\]

$n > 2 \quad \Rightarrow q_3 = q_5 = \ldots = 0$

For $s = -2$ we have the second solution,

\[
y_2(x) = \sum_{n=0}^{\infty} a_n x^{n-2} \quad q_0 \neq 0, \quad q_1 = 0
\]

The 3rd condition is in this case:
\[
q_n \left[ (n-2)(n-3) + 2n-5 \right] = -q_{n-2}
\]

\[
q_n = -\frac{q_{n-2}}{n^2-3n}
\]

**Thus, the solutions look like:**

\[
Y_1(x) = A \left[ x - \frac{x^3}{10} + \frac{x^5}{280} - \ldots \right]
\]

\[
Y_2(x) = B \left[ \frac{1}{x^2} + \frac{1}{2} - \frac{x^2}{8} + \ldots \right]
\]

The constants \(A\) and \(B\) must be determined from boundary conditions, as usual. When \(\lambda = 1/3\), \(Y_1(x)\) becomes the spherical Bessel function of order one; when \(B = -1\), \(Y_2(x)\) is the spherical Neumann function of order one.

As one might imagine, when the roots of the indicial equation are equal, this method fails to generate \(2\) independent solutions. What do we do in those cases, we consider next.

**Fech's Theorem**

This tells us when Frobenius' method will work. Write the 2nd-order differential equation as:

\[
y'' + f(x) y' + g(x) y = 0
\]
Then if:

\[
x f(x) \xrightarrow{x \to 0} 0
\]

(Fuchsin's conditions)

\[
x^2 g(x) \xrightarrow{x \to 0} \infty
\]

then the general solution consists of either:

1) two Frobenius series

2) one solution which is Frobenius, call it \( S_1(x) \)

Another solution which is \( S_2(x) \) is \( \ln x + S_2(x) \), where \( S_2(x) \) is another Frobenius series. [Or, you may want to try \( S_1(x) \ln x \) and solve for \( S_2(x) \)]

Case 2) happens only when the roots of the indicial equation are equal; or when they differ by an integer (but not always in this case!)

We now discuss an important example of the latter case.

**Bessel Functions**

Bessel equation in the usual standard form is

\[
x^2 y'' + x y' + (x^2 - \nu^2) y = 0
\]

\( \nu \) : any number, not necessarily integer

Dividing by \( x^2 \), we see that Bessel's theorem holds, i.e.

\[
x f(x) = 1 \quad \text{which is regular at} \ x > 0
\]

\[
x^2 g(x) = x^2 - \nu
\]

Again, we try Frobenius,

\[
y = \sum_{n=0}^{\infty} a_n x^{n+\nu}
\]

\[
x y' = \sum_{n=0}^{\infty} (n+\nu) a_n x^{n+\nu}
\]

\[
x^2 y'' = \sum_{n=0}^{\infty} (n+\nu)(n+\nu+1) a_n x^{n+\nu}
\]
\[
\sum_{n=0}^{\infty} \left[ (n+5)(n+5-1) + (n+5) + \frac{2}{3} \right] a_n x^{n+5} + x^2 \sum_{n=0}^{\infty} a_n x^{n+5} = 0
\]

Again, separate \(x^5, x^{5+1} \text{ (n=0, 1)}\)

\(\text{h=0:} \quad \left[ 5(s-1) + s - v^2 \right] a_0 = 0\)

\(\text{h=1:} \quad \left[ (s+1)s + s+1- v^2 \right] a_1 = 0\)

\(\text{h\geq 2:} \quad \left[ (n+5)(n+5-1) + (n+5) - v^2 \right] a_n + a_{n-2} = 0\)

Since \(a_0 \neq 0 \Rightarrow s^2 - v^2 = 0 \Rightarrow s = \pm v\)

The \(h=1\) condition gives: \(a_1 \left[ s_2 + 2s + 1 - v^2 \right] = a_1 \left[ 2s + 1 \right] = 0\)

Since \(2s+1 \neq 0\) when \(s = \pm v\) \(\Rightarrow a_1 = 0\)

The \(h\geq 2\) condition gives: \(\left[ (n+5)^2 - v^2 \right] a_n + a_{n-2} = 0\)

\(\Rightarrow a_n = -\frac{a_{n-2}}{(n+5)^2 - v^2}\)

\(\Rightarrow a_3 = a_5 = a_7 = \ldots = 0\)

\[a_n = -\frac{a_{n-2}}{n^2 + 2nv}\]

If \(s = +v\):

\[a_n = -\frac{a_{n-2}}{n^2 + 2nv}\]

This solution is known as Bessel function \(J_\nu(x)\) of order \(\nu\):

\[J_\nu(x) = \left(\frac{1}{2^\nu \nu!}\right) \sum_{n=0}^{\infty} \frac{(\frac{x}{2})^{n+\nu}}{\alpha_n a_0} x^{n+\nu} = \frac{1}{\alpha^n \nu!} \sum_{n=0}^{\infty} \frac{a_{2n}}{a_0} x^{2n+\nu}\]

where \(a_0 = \frac{1}{2^\nu \nu!}\) was chosen for convention; and \(\frac{a_n}{a_0} = 0\) for \(n\) odd, so only even terms contribute.

If \(s = -v\): The solution has the recurrence \(a_n = -\frac{a_{n-2}}{n^2 - 2nv}\)

\(a_1 = 0\)
\[ J_{\nu}(x) = \frac{1}{2^{\nu} \Gamma(\nu)} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n-\nu}}{a^n} \]

If \( \nu \) is not an integer, the general solution is

\[ y(x) = A J_{\nu}(x) + B J_{-\nu}(x) \]

However, if \( \nu \) is an integer, although the roots of indicial equation are not equal, they differ by an integer, and it turns out that \( J_{\nu}(x) \) and \( J_{-\nu}(x) \) are not independent solutions.

i.e. for \( \nu = m \) (integer) we have

\[ J_{-m}(x) = (-1)^m J_m(x) \]

So, the solution is actually obtained by using \( J_m(x), \ln x + S(x) \) and using the Gamma function, leading to the Neumann function \( N_m(x) \), which can be written in general as:

\[ N_{\nu}(x) = \frac{\cos(\pi \nu) J_{\nu}(x) - J_{-\nu}(x)}{\sin(\pi \nu)} \]

This looks weird, but it turns out to be useful. For \( \nu \) not integer, one can use either \( J_{\nu}(x) \) and \( J_{-\nu}(x) \) as solutions, or

\[ y(x) = A J_{\nu}(x) + B N_{\nu}(x) \]

Since \( N_{\nu}(x) \) is just a linear combination of \( J_{\nu} \) and \( J_{-\nu} \). For \( \nu \) integer, however,

\[ \cos(\pi \nu) J_{\nu}(x) = (-1)^\nu J_{-\nu}(x) \]

\[ \Rightarrow \cos(\pi \nu) J_{\nu}(x) - J_{-\nu}(x) = 0 \]

and also \( \sin(\pi \nu) = 0 \)
So, \( N(x) \) gives \( \frac{0}{0} \), when this indeterminate result is solved by l'Hopital rule, \( N(x) \) becomes a logarithmic times, \( J_n(x) \) something, as expected. So, \( N(x) \) is always the independent second solution of Bessel's equation.

Bessel functions are regular at the origin (since \( s = +r \)) and look like decaying sines and cosines:

On the other hand, \( N(x) \) are not regular at the origin.

We will come back to these functions when we discuss partial diff. equations.

Similarly to Legendre Polynomials, these guys obey recurrence relations, and are orthogonal, i.e.

\[
\begin{align*}
    J_{n-1}(x) + J_{n+1}(x) &= \frac{2n}{x} J_n(x) \\
    J_{n-1}(x) - J_{n+1}(x) &= \frac{2}{x} J_n(x) \\
    \text{etc}
\end{align*}
\]

And
\[
\int_0^a J_n \left( \alpha \sqrt{x} \right) J_n \left( \alpha \sqrt{r} \right) r \, dr = \frac{\alpha^2}{2} \left[ J_{n+1} (\alpha \sqrt{a}) \right]^2
\]

where \( \alpha \sqrt{m} \) is such that \( J_n (\alpha \sqrt{m}) = 0 \) for all \( m \); \( \alpha \sqrt{m} \) are the roots of \( J_n \) (see)