A matrix of order m x n, with m rows and n columns, is an array of numbers \( A_{ij} \):

\[
A = \begin{bmatrix}
   a_{11} & a_{12} & \ldots & a_{1n} \\
   a_{21} & a_{22} & \ldots & a_{2n} \\
   \vdots & \vdots & \ddots & \vdots \\
   a_{m1} & \ldots & \ldots & a_{mn}
\end{bmatrix}
\]

With some basic properties:

\( i \) Symmetry: \( A = B \) if and only if all elements are equal to each other, \( A_{ij} = B_{ij} \)

\( ii \) \( A + B = C \) if and only if \( A_{ij} + B_{ij} = C_{ij} \) (commutative)

\( \) \( (A + B) + C = A + (B + C) \) (associative)

\( iii \) Scalar multiplication:

If \( B = \alpha A \) then \( B_{ij} = \alpha A_{ij} \) i.e. each element gets multiplied

\( iv \) Multiplication of matrices:

\( A \cdot B = C \) means that

\[
C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}
\]

(i.e. we sum over index \( k \))

Which obeys:

\[
(AB)C = A(BC) \quad \text{(associative)}
\]

\[
A(B+C) = AB + AC \quad \text{(distributive)}
\]

But in general, \( AB \neq BA \)!! (not commutative)
\[
\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix}
\]

\[(AB)_{ij} = A_{ik}B_{kj} \Rightarrow i=j=1 \quad A_{11}B_{11} + A_{12}B_{12} = 1 \times 0 + 2 \times 3 = 6\]

\[
\begin{align*}
&i=1, \quad j=2 \\
&\quad A_{11}B_{12} + A_{12}B_{22} = 1 \times 1 + 2 \times 1 = 3 \\
&i=2, \quad j=1 \\
&\quad A_{21}B_{11} + A_{22}B_{21} = 0 \times 0 + 1 \times 3 = 3 \\
&i=2, \quad j=2 \\
&\quad A_{21}B_{12} + A_{22}B_{22} = 0 \times 1 + 1 \times 1 = 1
\end{align*}
\]

\[
A \quad B = \begin{bmatrix} 6 & 3 \\ 3 & 1 \end{bmatrix}
\]

On the other hand (try it) \[B \quad A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}\]

Then \[AB \neq BA\] - The difference between both \[AB - BA\] is called the commutator between \[A\] and \[B\]:

\[\begin{bmatrix} A, B \end{bmatrix} = AB - BA\]

In our example:
\[
\begin{bmatrix} A, B \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 0 & -6 \end{bmatrix}
\]

Commutators play a crucial role in quantum mechanics (meaning that quantities that don't commute cannot be measured simultaneously).

v) Determinant of square matrices \([n \times n]\)

Determinants of square matrices are defined in the usual way, e.g. for a 3x3 matrix \[\det A = \sum_{ijk} A_{ik}A_{ij}A_{jk}\]
It's easy to check that:

$$\det (A \pm B) = (\det A) (\det B)$$

however, $$\det (A + B) \neq \det A + \det B$$

vii) Trace of a Matrix

In any square matrix, the sum of the diagonal elements \((i = j)\) is called the trace, i.e.

$$\text{Tr} (A) \overset{\text{summation convention}}{=} \sum_{i} A_{ii} = A_{11} + A_{22} + A_{33} + \ldots + A_{nn}$$

One important property is that

$$\text{Tr} (AB) = \text{Tr} (BA)$$

even though $$AB \neq BA$$.

viii) Diagonal Matrices

A particular case is when a matrix has only non-zero elements on its diagonal, e.g. for a 3x3 matrix

$$A = \begin{pmatrix}
a_{11} & 0 & 0 \\
0 & a_{22} & 0 \\
0 & 0 & a_{33}
\end{pmatrix}$$

Note that

$$\text{Tr} (A) = a_{11} + a_{22} + a_{33}$$

Diagonal matrices commute in their product, $$AB = BA \iff A$$ and $$B$$ are diagonal.

The most important perhaps case of diagonal matrices is the unit matrix, where all diagonal elements are unity.
\[ I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \] or, in the 3x3 case \[ I = \begin{pmatrix} \delta_{ij} \end{pmatrix} \]

The elements of this matrix are the delta Kronecker that we defined before, i.e.

\[ I_{ij} = \delta_{ij} \]

and for 3x3, \[ \text{Tr} (I) = \delta_{ii} = 3 \] Obviously \[ A \cdot I = A \], for any matrix \( A \):

\[ (A \cdot I)_{ij} = \sum_k A_{ik} \delta_{kj} = A_{ij} = (A^i)_{ij} \]

**Rotation Matrices**

An important application of matrix multiplication is to study rotations. Consider the relation between coordinates \((x, y)\) of a point \(A\) and coordinates \((x', y')\) of the same point in a different system rotated (clockwise) an angle \(\theta\) with respect to the former:

\[
\begin{align*}
    x' &= x \cos \theta + y \sin \theta \\
y' &= -x \sin \theta + y \cos \theta
\end{align*}
\]

We can write this equation using matrices and vectors. Consider the vectors \( \vec{r}_A = (x_A, y_A) \) and \( \vec{r}_A' = (x_A', y_A') \) and the matrix

\[
R = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}
\]
Then we have that,

$$ F'_{R} = R \cdot F_{R} $$

or

$$ \begin{pmatrix} x'_{R} \\ y'_{R} \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x_{R} \\ y_{R} \end{pmatrix} $$

as can be easily checked using the rules of matrix multiplication,

$$ (F'_{R})_{i} = \sum_{j=1}^{2} R_{ij} (F_{R})_{j} \quad (\text{with } i = 1 \text{ the } x\text{-component}) $$

Note that $\det (R) = \cos^2 \theta + \sin^2 \theta = 1$, so a rotation matrix has a determinant $= 1$. Another transformation one might think about is a reflection, where

$$ \begin{cases} x'_{R} = x_{R} \\ y'_{R} = -y_{R} \end{cases} $$

that is, we invert the direction of the $y$-axis. Such a reflection has a matrix

$$ A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} $$

$$ \Rightarrow \det A = -1 \quad \text{So a reflection [in a single direction] has a determinant of } -1. \quad \text{Note that inverting } x\text{-axis as well gives again a } \det A = +1 \quad \text{this is in fact a rotation! You can convince yourself because you can continuously rotate } x-y \text{ axis until you get there; this is easy to see mathematically from the rotation matrix } R, \text{ for } \theta = 180^\circ, \cos \theta = -1, \sin \theta = 0 \text{ and } R = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. $$

Rotation matrices play an important role in physics: we will use them frequently.
Matrix inversion is accordingly,

\[ A \cdot A^{-1} = I \]

i.e. \( \sum_k A_{ik} A_{kj}^{-1} = \delta_{ij} \) in components.

The inverse of a square matrix can be found by the following formula:

\[ A^{-1} = \frac{C^T}{\det A} \]

where \( C \) is the matrix of cofactors, \( C_{ij} = (-1)^{i+j} M_{ij} \) where \( M_{ij} \) is the minor corresponding to element \( A_{ij} \). (Recall about this's from determinants.)

\( C^T \) is the transpose of \( C \), i.e. interchange rows and columns.

\[ (C^T)_{ij} = C_{ji} \]

For example, let's find the inverse of the rotation matrix \( R \)

\[ R = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \]

\[ \det R = 1 \]

\[
\begin{array}{c|c|c}
   & \text{Minor} & \text{Cofactor} \\
\hline
1 & 1 & R_{11} = \cos \theta, \quad (-1)^{1+1} \cos \theta = \cos \theta \\
1 & 2 & R_{12} = -\sin \theta, \quad (-1)^{1+2} (-\sin \theta) = \sin \theta \\
2 & 1 & R_{21} = \sin \theta, \quad (-1)^{2+1} \sin \theta = -\sin \theta \\
2 & 2 & R_{22} = \cos \theta, \quad (-1)^{2+2} \cos \theta = \cos \theta \\
\end{array}
\]

\[ \Rightarrow R^{-1} = \frac{1}{\det R} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \]

Remember we must transpose and cofactor.
We can easily check that
\[
R \quad R^{-1} = I, \quad \text{i.e.} \quad \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

In addition, it is easy to see this is the right answer, in order to undo a rotation by angle \( \theta \), we must do a rotation by angle \(-\theta\),

\[
R(-\theta) = \begin{pmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = R^{-1}
\]

\[
\cos(-\theta) = \cos \theta \\
\sin(-\theta) = -\sin \theta
\]

Now we can also understand why Cramer's rule works. Recall the example of linear equations we discussed before,

\[
\begin{align*}
    x + y &= 1 \\
    2y + z &= 1 \\
    x + z &= 0
\end{align*}
\]

We can write it in matrix form, by defining vectors

\[
\bar{F} = (x, y, z) \quad \bar{a} = (1, -1, 0)
\]

And matrix \( A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix} \)

\( \Rightarrow \) the set of linear equations \( A \bar{F} = \bar{a} \) can be written

\[
\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}
\]

with the obvious solution,
$$\overline{F} = A^{-1} \cdot \overline{a}$$

The question then is to find $A^{-1}$:

$$\det A = 1 \times \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} - 1 \times \begin{vmatrix} 0 & 1 \\ \lambda & 1 \end{vmatrix} + 0 \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} = 2 + 1 = 3$$

$$C = \begin{pmatrix} 2 & 1 & -2 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} \quad \Rightarrow \quad C^T = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & 1 \\ -2 & 1 & 2 \end{pmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{3} \begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & 1 \\ -2 & 1 & 2 \end{pmatrix} \quad \Rightarrow \quad \overline{F} = \frac{1}{3} \begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & 1 \\ -2 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

which is the solution obtained before by using Cramer's rule.

Cramer's rule works because

$$r_i = \frac{1}{3} [+2 +1] = 1$$

$$y = \frac{1}{3} [-1] = 0$$

$$z = \frac{1}{3} [-2 -1] = -1$$

An important rule that follows from this applies to systems of linear equations which are homogeneous (i.e. right hand side is equal to zero):
when \( \mathbf{A} \cdot \mathbf{F} = \mathbf{0} \)

If \( \mathbf{A} \) has an inverse, then

\[
\mathbf{F} = \mathbf{A}^{-1} \cdot \mathbf{F} < \mathbf{0} \Rightarrow x = y = z = 0
\]

\( \Rightarrow \) the solution is said to be trivial, \( \mathbf{F} = 0 \)

In order for the solution to be non-trivial (\( \mathbf{F} \neq 0 \)), we need \( \mathbf{A}^{-1} \) to be singular, that is, that \( \mathbf{A}^{-1} \) does not exist, so we need:

\[
\det \mathbf{A} = 0
\]

in this case \( \mathbf{A}^{-1} = \frac{\mathbf{C}^T}{\det \mathbf{A}} \) does not exist.

This is thus the requirement for finding non-trivial solutions to homogeneous linear equations.

Special Matrices

Here we define some simple operations on matrices,

\[
\begin{align*}
\text{Transpose} & : \quad (\mathbf{A}^T)_{ij} = \mathbf{A}_{ij} \\
\text{Complex conjugate} & : \quad (\mathbf{A}^*)_{ij} = (\mathbf{A}_{ij})^* \quad \text{(complex conjugate each element)} \\
\text{Hermitian conjugate} & : \quad (\mathbf{A}^H)_{ij} = \mathbf{A}_{ij}^* \quad \text{(complex conjugate + transpose)}
\end{align*}
\]

(\( ^* \): is a dagger)

From these operations it follows some nomenclature,

A matrix is called \textit{symmetric} if \( \mathbf{A} = \mathbf{A}^T \)

\textit{Antisymmetric} if \( \mathbf{A} = -\mathbf{A}^T \)
A modulus is called real if

\[ A = A^* \]
\[ A = -A^* \]
\[ A \cdot A^T = I \quad \text{i.e.} \quad A^T = A^{-1} \]
\[ A = A^T \]
\[ A \cdot A^T = I \quad \text{i.e.} \quad A^T = A^{-1} \]

pure imaginary
orthogonal
Hermitean
Unitary

For example, you can check that rotation matrices are orthogonal; you can use this to show that the scalar product between two vectors is the same in any coordinate system obtained from another by a rotation (as promised before).

Linear Combinations, Independence

A linear combination of vectors \( \overline{A}, \overline{B}, \overline{C} \) is \( \alpha \overline{A} + \beta \overline{B} + \gamma \overline{C} \), where \( \alpha, \beta, \gamma \) are numbers. A set of vectors is said to be independent if there is no way of finding \( \alpha, \beta, \gamma \) so that a linear combination of them is the zero vector.

We can form more complicated functions of vectors than that. An arbitrary function of a vector \( \overline{F} \) is said to be linear if

\[
\begin{align*}
\{ f(\overline{F}_1 + \overline{F}_2) &= f(\overline{F}_1) + f(\overline{F}_2) \quad \text{and} \\
f(\alpha \overline{F}) &= \alpha f(\overline{F})
\end{align*}
\]

The function can also be a vector as well, as explored in homework.