Eigenvectors and Eigenvectors

As we discussed for rotations, one can think of matrices as giving a linear transformation of a vector $\mathbf{F}$ to $\mathbf{F}'$, according to:

$$\mathbf{F}' = A \cdot \mathbf{F}$$

As we shall see in a moment, it is very useful to consider whether there are vectors that are "essentially" invariant under such a transformation, i.e.

$$A \cdot \mathbf{v} = \lambda \mathbf{v}$$  \hspace{1cm} (1)$$

Where the action of the matrix $A$ on $\mathbf{v}$ is to give back the same vector times a constant $\lambda$ - such vectors are called eigenvectors and the constant $\lambda$ eigenvalues. We will explore the physical meaning of these objects below.

To find a pair $\mathbf{v}, \lambda$ such that (1) implies,

$$B \cdot \mathbf{v} = (A - \lambda 1) \cdot \mathbf{v} = 0$$

We can now consider this a simple system of linear equations which is homogeneous. If the matrix $B$ were invertible, the only solutions would be the trivial $\mathbf{v} = B^{-1} \mathbf{0} = \mathbf{0}$, so the requirement is that $B$ be singular, that is:

$$\det B = \det (A - \lambda 1) = 0$$
This is known as the characteristic equation (recall homework 1). For an \( n \times n \) matrix, this has \( n \) solutions \( \lambda_n \) which constitute the set of eigenvalues. For example, take a \( 2 \times 2 \) matrix:

\[
A = \begin{bmatrix} 0 & -1 \\ -3 \frac{1}{2} & 1 \frac{1}{2} \end{bmatrix}
\]

The characteristic equation is:

\[
\text{det} \begin{bmatrix} -\lambda & -1 \\ -3 \frac{1}{2} & 1 \frac{1}{2} - \lambda \end{bmatrix} = \left| \begin{array}{cc} -\lambda & -1 \\ -3 \frac{1}{2} & 1 \frac{1}{2} - \lambda \end{array} \right| = -\lambda \left( \frac{1}{2} - \lambda \right) - \frac{3}{2} = 0
\]

\[\Rightarrow \lambda^2 - \frac{1}{2} \lambda - \frac{3}{2} = 0 \Rightarrow \lambda = \frac{1}{2} \pm \sqrt{\frac{1}{4} - 4 \left( -\frac{3}{2} \right)} = \frac{1}{4} \pm \frac{1}{2} \sqrt{25} = \frac{1}{4} \pm \frac{1}{2} \frac{5}{2}
\]

\[\Rightarrow \lambda = \frac{1}{4} \pm \frac{5}{4} \Rightarrow \lambda_1 = 3 \frac{1}{2}, \quad \lambda_2 = -1
\]

The eigenvectors are found by putting this back into (1). For \( \lambda = 3 \frac{1}{2} \) we have:

\[
\begin{bmatrix} 0 & -1 \\ -3 \frac{1}{2} & 1 \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{3}{2} \begin{bmatrix} x \\ y \end{bmatrix}
\]

Which gives the system of equations:

\[
\begin{cases}
-\frac{3}{2} x + y = \frac{3}{2} y \\
-\frac{3}{2} x + \frac{3}{2} y = -\frac{3}{2} x = y
\end{cases}
\]

Notice both equations are actually the same, as it should be, i.e., if a vector \( \vec{v} \) is an eigenvector, any scalar multiple \( k \vec{v} = \vec{v'} \) is an eigenvector as well, so \( x \) and \( y \) should only be determined up to
So we have

$$\vec{v}_1 = (x, -\frac{3}{2}x) = x (1, -\frac{3}{2})$$

we can choose the overall constant $x$ as we please, usually it is chosen so that the norm of $\vec{v}_1$ is unity,

$$v_1^2 = x^2 \sqrt{1^2 + (-\frac{3}{2})^2} = 1 \Rightarrow x = \frac{1}{\sqrt{1 + \frac{9}{4}}} = \sqrt{\frac{13}{4}}$$

For $\vec{v}_2$, we have similarly,

$$\begin{bmatrix} 0 & -1 \\ -\frac{3}{2} & 1/2 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -\frac{3}{2}x + \frac{1}{2}y \end{pmatrix} \Rightarrow \begin{cases} -y = -x \\ \frac{3}{2}x + \frac{1}{2}y = -y \Rightarrow \frac{3}{2}x - \frac{3}{2}y \end{cases}$$

$$\Rightarrow \vec{v}_2 = (x, x) = x (1, 1)$$

$$v_2^2 = x^2 \sqrt{1^2 + 1^2} = 1 \Rightarrow x = \frac{1}{\sqrt{2}}$$

So the eigenvalues and eigenvectors are:

$$\begin{cases} \lambda_1 = \frac{3}{2} \\ \vec{v}_1 = \sqrt{\frac{13}{4}} (\frac{1}{2}, -\frac{3}{2}) \end{cases}$$

$$\begin{cases} \lambda_2 = -1 \\ \vec{v}_2 = \frac{1}{\sqrt{2}} (1, 1) \end{cases}$$

The special case of a (real) symmetric matrix is important. In this case, it is easy to show that the eigenvectors for eigenvalues that are different are orthogonal.

Consider two such eigenvectors, corresponding to $\lambda_1$ and $\lambda_2$,
\[ \begin{align*}
A \cdot \vec{v}_1 &= \lambda_1 \vec{v}_1 \\
A \cdot \vec{v}_2 &= \lambda_2 \vec{v}_2
\end{align*} \]

To show that they are orthogonal, we take their scalar product, e.g. take a look at

\[
\bar{v}_2 \cdot (A \cdot \bar{v}_1) = (\bar{v}_2)_i (A \cdot \bar{v}_1)_i = (\bar{v}_2)_i \lambda_1 (\bar{v}_1)_i = \lambda_1 \bar{v}_2 \cdot \bar{v}_1
\]

but we can rewrite this as

\[
(A \cdot \bar{v}_1)_i = (\bar{v}_2)_i \lambda_1 (\bar{v}_1)_i = (\bar{v}_2)_i \lambda_2 (\bar{v}_1)_i = (A \cdot \bar{v}_1)_j
\]

Now, from (A) and (B) we have

\[
\lambda_1 \bar{v}_1 \cdot \bar{v}_2 = \lambda_2 \bar{v}_1 \cdot \bar{v}_2
\]

\[
\Rightarrow (\lambda_1 - \lambda_2) \bar{v}_1 \cdot \bar{v}_2 = 0 \Rightarrow \lambda_1 = \lambda_2
\]

Since eigenvectors can be normalized to unit vectors, we have that for a real symmetric matrix

\[
\forall \ v^m, v^n \ = \ \delta_{nm} \quad m, n = 1, \ldots, N
\]

for the \( n \)th eigenvector corresponding to \( \lambda_m \), and on \( n \)th for \( \lambda_n \).

The eigenvectors \( \bar{v}^m \) can be used to define the columns of a matrix \( \Psi \):

\[
\Psi = \begin{bmatrix}
\bar{v}^1 & \bar{v}^2 & \cdots & \bar{v}^m & \cdots & \bar{v}^N
\end{bmatrix}
\]

(\( N \times N \) matrix)
whose elements are $V_{ij} = V_{k}^{(i)}$

(i.e. the $i$th component of the $j$th eigenvector. It is easy to see that such a matrix is orthogonal

$V^T V = 1$

where $T$ means transposed - we have

$(V^T V)_{ij} = (V^T)_{ik} V_{kj}$

$= V_{ki} V_{kj} = V_{k}^{(i)} V_{k}^{(j)} = \delta_{ij}. \delta_{ij} = \delta_{ij} = (1)_{ij} \check{\sqrt{}}$

As you will prove in HMK#2, problem 2, for $N$ vectors that are orthonormal it follows the so-called completeness relation

$I = \sum_{n=1}^{N} \psi^{(n)}(x) \psi^{(n)T}(x)$

(4)

where for a vector say $\psi^{(1)} = (x_1, y_1)$ its transpose is the $\psi^{(1)T} = (x_1, y_1)$

The $n=1$ term in the above sum is then the matrix

$[x_1^2, x_1 y_1, x_1 y_1, y_1^2]$

We can multiply equation (4) by any vector $\vec{p}$ to get

$\lambda \cdot \vec{p} = \vec{p} = \sum_{n=1}^{N} \psi^{(n)}(x) \psi^{(n)T}(x) \cdot \vec{p} = \sum_{n=1}^{N} \psi^{(n)}(x) \cdot (\psi^{(n)}(x) \cdot \vec{p})$

this is the scalar product (or inner product) $\psi^{(n)T} \cdot \vec{p}$

where we take rows of $\psi^{(n)}$ and multiply to columns of $\vec{p}$ (only a row and column in this case). Since vectors can be thought of rows or
Then we have the remarkable result,

\[
\bar{p} = \sum_{n=1}^{N} \bar{v}^{(n)} \cdot (\bar{v}^{(n)} \cdot \bar{p})
\]

[Emphasis: an error in class]

Now we can easily find the effect of the matrix \( A \) on any vector \( \bar{p} \):

\[
A \cdot \bar{p} = \sum_{n=1}^{N} \lambda_n \bar{v}^{(n)} \cdot (\bar{v}^{(n)} \cdot \bar{p})
\]

\[
\Rightarrow A \cdot \bar{p} = \sum_{n=1}^{N} \lambda_n \bar{v}^{(n)} \cdot (\bar{v}^{(n)} \cdot \bar{p})
\]

Compare this equation to \((\#)\), only effect is to add \( \lambda_n \) factor to each term. We can rewrite this equation as

\[
A \cdot \bar{p} = \left( \sum_{n=1}^{N} \lambda_n \bar{v}^{(n)} \bar{v}^{(n)\top} \right) \cdot \bar{p}
\]

Since this is true for any \( \bar{p} \), we have that

\[
A = \sum_{n=1}^{N} \lambda_n \bar{v}^{(n)} \bar{v}^{(n)\top}
\]

We can rewrite this in components as,

\[
A_{ij} = \sum_{n=1}^{N} \lambda_n \bar{v}_i^{(n)} \bar{v}_j^{(n)}
\]

\[
= \sum_{n=1}^{N} \lambda_n \bar{v}_{in} \bar{v}_{jn} = \sum_{n=1}^{N} \lambda_n \bar{v}_{in} \bar{v}_{jn}\]

[Note: \( \bar{v}^{(n)} \bar{v}^{(m)\top} \) is column times row, so it is not a scalar product, it is a matrix product. \( \bar{v}^{(n)\top} \bar{v}^{(n)} = 1 \) is the scalar product]
\[ A_{ij} = V_{ik} \Sigma_{kl} V_{lj} \]

where \( \Sigma_{kl} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \) are the elements of the diagonal matrix \( \Sigma \).

Thus, the matrix of the eigenvectors \( V \), can be used to connect the matrix \( A \), with its diagonal version \( \Sigma \). Then we can diagonalize \( A \) by multiplying by \( V^T \) and \( V \) on left and right,

\[ V^T A V = \begin{bmatrix} V^T & 0 & \cdots & 0 \\ 0 & V^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & V^T \end{bmatrix} \begin{bmatrix} \Sigma & \cdots & 0 \\ \cdots & \ddots & \cdots \\ 0 & \cdots & \Sigma \end{bmatrix} = \Sigma = \text{diag}(\lambda_1, \ldots, \lambda_n) \]

What is the meaning of \( V \)? Let's multiply \( V \) times the unit vector along the first direction (say \( x \)-axis) \( \hat{e}_1 = (1, 0, \ldots, 0) \).

We have:

\[ V \cdot \hat{e}_1 = \begin{bmatrix} \vec{v}^{(1)} \\ \vdots \\ \vec{v}^{(n)} \end{bmatrix} (1, 0, \ldots, 0) = \vec{v}^{(1)} \]

Similarly,

\[ V \cdot \hat{e}_2 = \vec{v}^{(2)} \ldots \quad V \cdot \hat{e}_n = \vec{v}^{(n)} \]
Next, the matrix $W$ relates the $(x_1, y_1, z_1, \ldots)$ axes along the eigenvectors of $A$!

$W$ can indeed be regarded as a rotation matrix, since it is orthogonal, with $\det W = 1$.

Thus, to diagonalize a matrix, one needs to find its eigenvectors.

**NORMAL MODES**

An important application of eigenvalues and eigenvectors is to find the characteristic oscillations of physical systems. We shall consider a simple example [(12.6 of Smieda)] where we have 3 masses connected by 2 springs.

The equations of motion are

\[
\begin{align*}
    m \ddot{x}_1 &= k(\gamma_2 - \gamma_1) \\
    m \ddot{x}_2 &= -k(\gamma_2 - \gamma_1) + k(\gamma_3 - \gamma_2) \\
    m \ddot{x}_3 &= -k(\gamma_3 - \gamma_2)
\end{align*}
\]

(I'm assuming that rest length for springs is $l=0$)

We look for solutions to these equations where $x_i$ depend on time as $e^{-i\omega t}$ [i.e., $\cos \omega t$ or $\sin \omega t$, depending on initial conditions]. Then we have $\ddot{x}_i = -\omega^2 x_i$, and we can write

\[ A \cdot \vec{x} = \frac{m\omega^2}{k} \vec{x} \]

where $A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$ and $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

So $m\omega^2$ are eigenvalues of $A$!
The normal modes are the characteristic pattern of oscillations. These oscillations are found by looking for the eigenvalues of $A$; that is, this is so because

$$(A - \frac{m \omega^2}{k} \mathbb{1} \mathbb{1}) \mathbf{x} = 0$$

will only have non-trivial solutions for eigenvalues $\lambda$ so that $\det (A - \lambda \mathbb{1}) = 0$ (where $\lambda = \frac{m \omega^2}{k}$)

$$\Rightarrow \det \begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{bmatrix} = (1-\lambda) [(2-\lambda)(1-\lambda)-1] + 1 (\lambda-1) = 0$$

$$\Rightarrow (\lambda^2 - 3\lambda + 1)(\lambda - 1) = \lambda(\lambda - 3)(\lambda - 1) = 0 \Rightarrow \lambda = \left\{ \frac{1}{3}, 1 \right\}$$

Therefore, the eigen frequencies are $\omega_1 = 0$, $\omega_2 = \sqrt{\frac{k}{m}}$, $\omega_3 = \sqrt{\frac{3k}{m}}$

At these frequencies, the system will oscillate freely. Let's figure out the eigen vectors, which give the displacement pattern for each eigenfrequency. They are [126.1d in Snider]

$$\mathbf{v}^{(1)} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}^{(2)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{v}^{(3)} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

as you can easily derive (try it!). So, we see that for $\omega_1 = 0$, we have a displacement pattern for each mass.

$$\uparrow \rightarrow \uparrow \rightarrow \uparrow \rightarrow$$

So, the 3 masses move in the same direction with the same
amplitude, there is no oscillation at all! Only the center of mass moves. That's why \( w = 0 \), independent of \( k/m \).

For \( w_2 = \sqrt{1/v_m} \) we have the pattern

\[
\begin{array}{ccc}
1 & \rightarrow & 0 \\
\uparrow & & \downarrow \\
0 & \rightarrow & 0 \\
\uparrow & & \downarrow \\
0 & \rightarrow & 1 \\
\end{array}
\]

Note that the center of mass doesn't move. The masses at the ends move only, and they do so in a single mass harmonic motion (since central mass doesn't move) - That's why frequency is the usual single-mass oscillation \( w^2 = k/v_m \).

For the last eigenmode \( w_3^2 = 3k/v_m \) we have the pattern,

\[
\begin{array}{ccc}
1 & \rightarrow & -2 \\
\uparrow & & \downarrow \\
-2 & \rightarrow & 0 \\
\uparrow & & \downarrow \\
0 & \rightarrow & 1 \\
\end{array}
\]

Note again that COM doesn't move (all this motion is in \( \vec{r}^{(n)} \)).

This has a higher frequency, as expected because the springs get more compressed.

The physical importance of these eigenmodes is that any displacement vector \( \vec{d} \), arbitrary, can be decomposed into eigenmodes, since all 3 are orthogonal, we have

\[
\vec{d} = \sum_{n=1}^{3} (\vec{r}^{(n)} \cdot \vec{d}) \left( \vec{r}^{(n)} \right)
\]

(see eq. # in page 6)

This says how much COM motion \((n=1)\) there is,
how much single-mass oscillation \((n=2)\) there is,
how much high-frequency oscillation \((n=3)\) there is.

This is also useful to describe forced oscillations, see homework 2 problem 10 [12.6 i in Smiede]