Matrices and Coordinate Transformations

Let's determine the transformation properties of matrices under coordinate transformations, which we take to be orthogonal (or unitary, since they are real operators).

If a vector $\vec{x}$ is mapped to $\vec{y}$ under the action of some matrix $A$, $\vec{y} = A \cdot \vec{x}$.

We want to know how things change when seen in a different coordinate system. In this new coordinate system, the new components of vector $\vec{x}$ are given by $\vec{x}'$: $\vec{x}' = C \cdot \vec{x}$

where $C$ is the orthogonal matrix $(C^{-1} = C^T)$ which describes the coordinate transformation. Similarly, $\vec{y}' = C \cdot \vec{y}$.

Then, we have: $\vec{y}' = C \cdot \vec{y} = C \cdot A \cdot \vec{x} = C \cdot A \cdot C^T \vec{x}'$

So, if we define the matrix, $A' = C \cdot A \cdot C^T$ \hfill (1)
we have that

\[ \tilde{y}' = A' \cdot \tilde{x}' \]

Which is the same equation as before, but now expressed in the new coordinate system. Therefore matrices transform as \((\#)\) under a coordinate transformation that transforms vectors as \((\Omega)\).

Recall that we saw this already when we studied the coordinate transformation that went from \(e^n\) to eigenvectors, \((\#)\) was obtained for the matrix in the new coordinate system (i.e., where the matrix was diagonal).

We can now rewrite the transformation properties of vectors and matrices under orthogonal transformations, in terms of components

\[
\begin{align*}
X'_{i} &= C_{ij} \cdot x_{j} \\
A'_{ij} &= C_{ik} \cdot A_{kl} \cdot (CT)_{lj} = C_{ik} \cdot C_{lj} \cdot A_{kl}
\end{align*}
\]

Note that we are using here the summation convention.

Note the structure of these equations: i) There are as many \(C's\) as indices. ii) The first index of each \(C\) matches the indices found on the object in the new coordinate system (\(\tilde{x}'\) or \(A'\)). iii) The second index in \(C's\) are contracted (summed over) with two of the object in the old coordinate system.

Note: for quantities with no indices, scalars, there is no \(C\) acting on it, so scalars are the same in any coordinate system.
For example, take the scalar product between 2 vectors $\mathbf{x}$ and $\mathbf{y}$:

$$\mathbf{x} \cdot \mathbf{y} = x_i y_i = \mathbf{C}_{ij}^{-1} x^i C_{ik}^{-1} y^k = \frac{C_{ji} C_{ik}}{\delta_{jk}} x^j y^k = x'_j y'_k$$

$$\Rightarrow \mathbf{x} \cdot \mathbf{y} = \mathbf{x}' \cdot \mathbf{y}'$$

As expected, the scalar product is a scalar, it has the same value in all coordinate systems; no C's involved. Obviously, $x^2$ is also a scalar ($= \mathbf{x} \cdot \mathbf{x}$).

These properties under transformations are what we need to define tensors.

**Tensors**

A physical quantity $T_{i_1 i_2 \ldots i_n}$ is called a tensor of rank n when it transforms under orthogonal transformations $C$ according to,

$$T_{i_1 i_2 \ldots i_n}' = C_{i_1 j_1} C_{i_2 j_2} \ldots C_{i_n j_n} T_{j_1 j_2 \ldots j_n}$$

So, vectors are tensors of rank 1, and the rule for matrices above corresponds to this for $n=2$.

Note that tensors are defined in terms of their transformation properties. Not every vector or matrix is a tensor, it depends on how they transform under coordinate transformations.

We will see examples of tensors during this course. An interesting application of tensor calculus is special relativity as we now discuss.
We now consider the transformation between inertial systems, that is, systems which move with respect to each other at constant velocity $v$.

In Newtonian physics, such transformation is given by a so-called Galilean transformation:

\[
\begin{align*}
    x' &= x - vt \\
y' &= y \\
z' &= z \\
t' &= t
\end{align*}
\]

where the system $S'$ is moving with respect to system $S$, at velocity $v$ in the $x$-direction. Note that perpendicular directions do not change, and also time is absolute, $t = t'$; time intervals are the same in $S$ and $S'$.

Newtonian mechanics is written in vectorial form, as we discussed already, and all inertial systems are equivalent, e.g., the 2nd law is valid in all inertial systems,

in $S$: $\vec{F} = m \ddot{\vec{a}}$ or $\vec{F}' = m \ddot{\vec{a}'}$ (in $S'$)

One problem with the Galilean transformations (\#), as became evident in the late 1800's was that it leads to the addition of velocities,

\[
\begin{align*}
    u' &= \frac{dx'}{dt'} = \frac{dx}{dt} = \frac{d\xi}{dt} - v = u - v
\end{align*}
\]

So, if we see something moving at velocity $u$ in $S$, it's moving at
u' = u - v in system S'

The question arose whether light does actually have this additive property. If so, one could measure the speed of light when the earth moves in one direction, and then six months later when it moves in the opposite direction, and the difference will tell us how fast is the earth moving with respect to the frame where the speed of light is c (which was then called 'ether'). Such a measurement (e.g. Michelson-Morley experiment, and others) demonstrated that the speed of light is independent of the motion of the source.

Einstein based his special relativity on the postulates:

i) All physical laws are the same in any inertial system (same as Newton's case

ii) The speed of light has an absolute value independent of inertial system = c \approx 300,000 km/s

This second postulate leads to very different transformation laws as we now derive. We look for linear transformations that connect (x,t) and (x',t') - for simplicity we assume that

y = y', z = z'; this can be shown to hold by more general arguments.

A linear transformation is

x = a \cdot x' + b \cdot t'

But we should also have

x' = a \cdot x - b \cdot t

Since viewed from the other system, things should have the same form.
due to postulate i), when time is running backwards.

In fact \( x = 0 \) describes the motion of the origin of coordinates of \( S' \)

\[
0 = ax' + bt' \Rightarrow \frac{b}{a} = v
\]

and similarly \( x' = 0 \) is the motion of origin of \( S' \) seen from \( S \),

\[
0 = ax - bt \Rightarrow \frac{b}{a} = vt \Rightarrow \frac{b}{v} = t \checkmark
\]

Now we need to use the 2nd postulate, ii), that speed of light is the same in \( S \) and \( S' \). We assume that when both origins coincide a light bulb is turned on. The signal is described by \( S \) and \( S' \) as:

\[
\begin{align*}
\begin{cases}
x = ct \\
x' = ct'
\end{cases}
\end{align*}
\]

\[
\Rightarrow \quad c t = a \left( ct' + v t' \right)
\]

\[
\Rightarrow \quad c t' = a \left( ct - v t \right) \Rightarrow c t' = a \left( c - v \right) t
\]

\[
\Rightarrow \quad c t = a \left[ a \left( c - v \right) t + \frac{v^2}{c^2} a \left( c - v \right) t \right] = a^2 \left( c - v \right) \left( 1 + \frac{v}{c} \right) t
\]

\[
\Rightarrow \quad \frac{a^2}{c} = \frac{c-v}{c} \left( 1 + \frac{v}{c} \right) = \left( 1 - \frac{v}{c} \right) \left( 1 + \frac{v}{c} \right) = 1 - \beta^2 \quad \beta = \frac{v}{c}
\]

\[
\Rightarrow \quad a = \gamma = \frac{1}{\sqrt{1 - \beta^2}}
\]

Therefore, the Lorentz transformation becomes,

\[
\begin{align*}
\begin{cases}
x = \gamma (x' + vt') \\
c t = \gamma (ct' + \beta x')
\end{cases}
\end{align*}
\]

or

\[
\begin{align*}
\begin{cases}
x' = \gamma (x - vt) \\
c t' = \gamma (ct - \beta x)
\end{cases}
\end{align*}
\]

Of course, the inverse transformation follows by taking \( v \rightarrow -v \)

\( (\beta \rightarrow -\beta) \) as we discussed before, at times (when moving)
Note that when $v \ll c, \beta \ll 1 \Rightarrow \gamma \approx 1$

\[
\begin{cases}
    x' = x - vt \\
    ct' = ct
\end{cases}
\]

as for Galilean transformation

[so, for small speeds, relativity reduces to uniform case.]

The transformation (7) is known as Lorentz transformations. A lot of interesting physics follows from these (e.g., relativity of simultaneity) that hopefully you will soon see in future courses.

Now we can see how this new transformations can be applied to tensors. We define a 4-vector

\[
\mathbf{x}^\mu = \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix}
\]

Note that 4th component is imaginary, $ict - 1 + s$ magnitude is

\[
|\mathbf{x}|^2 = x^2 + y^2 + z^2 - c^2 t^2
\]

The Lorentz transformation can be written as:

\[
\mathbf{\tilde{x}} = L \cdot \mathbf{x}
\]

where

\[
L = \begin{bmatrix}
    1 & 0 & 0 & \beta x \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    -\beta y & 0 & 0 & 1
\end{bmatrix}
\]

So, $L$ is a tensor of rank 1. You can check that $L^T = L^{-1}$, since this switches $\beta \Rightarrow -\beta$ ($v \Rightarrow -v$). Also, similarly to coordinate transformations, Lorentz transformations conserve the scalar product between 4-vectors,

\[
\mathbf{x} \cdot \mathbf{\tilde{x}} = \mathbf{x}^\dagger \mathbf{\tilde{x}}
\]
and in particular, the magnitude of a 4-vector, 

\[ r^2 = x^2 + y^2 + z^2 - ct^2 = x'^2 + y'^2 + z'^2 - c't'^2 \]

[Note that this implies that the light front in the example alone, propagates as a velocity = c spherical front in both S and S', as required by the 2nd postulate.]

An interesting application of tensors in special relativity, is that electric (E) and magnetic (B) fields are mixed by lorentz transformations. For example, they transform as a tensor of rank 2,

\[
\mathbf{F}' = \mathbf{H} = \begin{bmatrix}
0 & B_z & -B_y & -i \frac{E_z}{c} \\
-B_z & 0 & B_x & -i \frac{E_x}{c} \\
B_y & -B_x & 0 & -i \frac{E_y}{c} \\
i \frac{E_x}{c} & i \frac{E_y}{c} & i \frac{E_z}{c} & 0
\end{bmatrix}
\]

So under a lorentz transformation, \( F'_{ij} = \mathbf{H}_{ik} \mathbf{L}_{kj} \mathbf{H}_{ml} \)

\[
\begin{align*}
E'_x &= \gamma (E_x - \nu B_y) \\
E'_y &= \gamma (E_y + \nu B_x) \\
E'_z &= E_z \\
B'_x &= \gamma (B_x + \beta E_y/c) \\
B'_y &= \gamma (B_y - \beta E_x/c) \\
B'_z &= B_z
\end{align*}
\]

Say we have a charge \( q \) at rest in \( x=0 \) at \( S \), and \( E=0, \ B=0 \).

In \( S' \), \( q \) is moving so it will have a force due to magnetic field

\[ F' = -q \left( \nu \mathbf{x} \times \mathbf{B}' \right) \approx -q \mathbf{u} \times \mathbf{B} \]

but, there is an electric field \( E' = \nu \mathbf{x} \mathbf{B} \), so this additional force cancels out for \( E' = 0 \Rightarrow \mathbf{F}' = 0 \), as expected, so it should be.