CONSERVATIVE FIELDS: POTENTIAL THEORY

The definitions here derive from the familiar situation in particle mechanics, where a force is said to be conservative if the work done by displacing a particle from point A to point B is independent of the path from A to B:

\[ \text{Work} = \oint_{\gamma} \mathbf{F} \cdot d\mathbf{r} \]

Remember that when we discussed gradients, we showed that the gradient of a function \( \nabla f \) has this property - let's show that's the case.

If integrals by path 1 and 2 give same answer, I can construct a curve \( \gamma \) which goes on path 1 from A to B, and comes back from B to A on - path 2, so:

\[ \oint_{\gamma} \mathbf{F} \cdot d\mathbf{r} = \oint_{\text{path 1}} \mathbf{F} \cdot d\mathbf{r} + \oint_{\text{-path 2}} \mathbf{F} \cdot d\mathbf{r} \]

Since work is independent of path, it changes sign when I interchange A and B, we have

\[ \text{Work} = \oint_{A \rightarrow B} \mathbf{F} \cdot d\mathbf{r} = \phi(B) - \phi(A) \]

If we choose A and B very close together, we have infinitesimal relation:

\[ \mathbf{F} \cdot d\mathbf{r} = - (\phi(B) - \phi(A)) = d\phi = - \nabla \phi \cdot d\mathbf{r} \]
Since gradient gives change in function after displacement $d\vec{r}$, then we have:

$$ \nabla \phi = -\vec{F} $$

Obviously, the converse is also true: \( \frac{\partial F}{\partial x} \cdot d\vec{r} = 0 \iff \vec{F} = -\nabla \phi \) (for simply connected region).

Note that \( \nabla \times \vec{F} = -\nabla \times (\nabla \phi) = 0 \) - This also follows from Stokes's theorem:

\[
\oint_{\Gamma} \vec{F} \cdot d\vec{r} = 0 = \iint_{\partial \Gamma} (\nabla \times \vec{F}) \cdot d\vec{S}
\]

Then we have that for a conservative field $\vec{F}$,

"Work is independent" of path:

$$ \oint_{\Gamma} \vec{F} \cdot d\vec{r} = 0 $$

Stokes's Theorem

\( \nabla \times (\nabla \phi) = 0 \)

Therefore, any conservative field can be written as the gradient of a scalar potential $\phi$.

The divergence of the form $\vec{F}$ (or any conservative vector field) can be written as

$$ \nabla \cdot \vec{F} = -\nabla \cdot (\nabla \phi) = -\nabla^2 \phi $$

where $\nabla^2$ is known as the **Laplacian**. In cartesian coordinates,

$$ \nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) $$

\[ \Rightarrow \nabla \cdot (\nabla \phi) = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \left( \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \]
\[ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \]

In cylindrical coordinates \((r, \theta, z)\) you derived in HWB that

\[ \nabla \cdot \mathbf{\phi} = \frac{\partial \phi}{\partial r} \mathbf{\hat{r}} + \frac{1}{r} \frac{\partial \phi}{\partial \theta} \mathbf{\hat{\theta}} + \frac{\partial \phi}{\partial z} \mathbf{\hat{z}} \]

and

\[ \nabla \cdot \mathbf{\nabla} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial \phi}{\partial r}) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (r^2 \sin \theta \frac{\partial \phi}{\partial \theta}) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \phi}{\partial \theta^2} = \nabla^2 \phi \]

In spherical coordinates \((r, \theta, \phi)\) you also derived in HWB that

\[ \nabla \cdot \mathbf{\phi} = \frac{\partial \phi}{\partial r} \mathbf{\hat{r}} + \frac{1}{r} \frac{\partial \phi}{\partial \theta} \mathbf{\hat{\theta}} + \frac{1}{r \sin \theta} \frac{\partial \phi}{\partial \phi} \mathbf{\hat{\phi}} \]

and

\[ \nabla \cdot \mathbf{\nabla} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial \phi}{\partial r}) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (r^2 \sin \theta \frac{\partial \phi}{\partial \theta}) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \phi}{\partial \phi^2} = \nabla^2 \phi \]

Going back to \(E\), we see that the Laplacian of \(\phi\) acts as a source or sink (depending on whether \(\nabla^2 \phi < 0\) or \(> 0\)) for \(E\):

\[ \nabla \cdot \mathbf{E} = -\nabla^2 \phi \]

What about \(\nabla \times \mathbf{E}\)? Since \(\nabla \times \mathbf{E} = 0\) when \(\mathbf{E}\) conserves, there is no vorticity in \(\mathbf{E}\), or no circulation. The flow of \(\mathbf{E}\) is just a pure divergence, sinks and sources, but no "rotational motion".

There are cases in which the opposite is true, i.e. flows...
which have non-zero curl or rotation, but zero divergence.

[we shall see in a moment that divergence and curl specify completely a vector field, so each type of flow is a "basis" for any more general situation].

If \( \nabla \cdot \mathbf{B} = 0 \) for a vector field \( \mathbf{B} \) (such as the magnetic field \( \mathbf{B} \)) then from \( \nabla \cdot (\mathbf{B} \times \mathbf{A}) = 0 \) we see that we can write

\[
\mathbf{B} = \nabla \times \mathbf{A} \quad \iff \quad \nabla \cdot \mathbf{B} = 0
\]

\( \mathbf{A} \) is said to be the vector potential. 

Note that if we are given \( \mathbf{B} \), we can find an infinite number of \( \mathbf{A} \)'s such that \( \mathbf{B} = \nabla \times \mathbf{A} \). Indeed if \( \mathbf{A}_1 \) is such that \( \mathbf{B} = \nabla \times \mathbf{A}_1 \) then

\[
\mathbf{A}_2 = \mathbf{A}_1 + \nabla \phi
\]

satisfies \( \nabla \times \mathbf{A}_2 = \nabla \times \mathbf{A}_1 + \nabla \times (\nabla \phi) = \mathbf{B} \).

So, we say that \( \mathbf{A} \) is determined up to a gradient.

To summarize we have two types of "vector flows",

1) \( \nabla \cdot \mathbf{A} = 0 \) \( \nabla \times \mathbf{A} = \mathbf{0} \)

\textit{Rotational or Solenoidal}

\( \Rightarrow \mathbf{A} = \nabla \times \mathbf{A} \) for some vector potential \( \mathbf{A} \)

2) \( \nabla \cdot \mathbf{V} = 0 \) \( \nabla \times \mathbf{V} = \mathbf{0} \)

\textit{Divergence or Potential or "Conservative"}

\( \Rightarrow \mathbf{V} = -\nabla \phi \) for some scalar potential \( \phi \)
Magnetic fields (in static case) are of rotational character. Electric fields (in static case) and gravity are of potential character.

Now we show a remarkable theorem, which says that any vector field can be written in terms of its divergence and curl.

This is important because we can think locally (at near any point) that the flow is a linear superposition of rotational and potential flows.

**HELMHOLTZ THEOREM**

First consider the following statement:

i) "A vector is uniquely specified by giving its divergence and its curl within a region and its normal component over the boundary."

Proof: Let's assume

$$\nabla \cdot \vec{V}_i = 0$$

$$\nabla \times \vec{V}_i = \vec{\omega}$$

and that $$\vec{V}_i \cdot \hat{n} = V_{ni}$$ at the boundary. If there is a second vector \(\vec{V}_2\) satisfying these same conditions, we want to see that $$\vec{V}_2 = \vec{V}_1$$, therefore proving that it is unique. Let

$$\vec{d} = \vec{V}_1 - \vec{V}_2$$

Then we have by definition:

$$\begin{cases} 
\nabla \cdot \vec{d} = 0 \\
\nabla \times \vec{d} = 0 \\
\vec{d} \cdot \hat{n} = 0 \text{ at boundary}
\end{cases}$$

Since \(\vec{d}\) is irrotational we may write,

$$\vec{d} = -\nabla \phi$$

$$\Rightarrow \nabla \cdot \vec{d} = -\nabla^2 \phi = 0 \quad \text{(Laplace's equation)}$$
To show that \( \vec{a} \) is the zero vector (and thus \( \vec{v}_1 = \vec{v}_2 \)) we will use Green's theorem:

\[
\iint_S \phi_1 \nabla \phi_2 \cdot d\vec{S} = \iint_V (\phi_1 \nabla \phi_2 + \phi_2 \nabla \phi_1) \cdot d\vec{S}
\]

This can be easily shown from Gauss' theorem. Since

\[
\nabla \cdot (\phi_1 \nabla \phi_2) = \nabla \phi_1 \cdot \nabla \phi_2 + \phi_1 \nabla^2 \phi_2
\]

we have from Gauss that

\[
\iiint_V (\phi_1 \nabla^2 \phi_2 + \phi_2 \nabla^2 \phi_1) \, d^3 \vec{r} = \iiint_V \nabla \cdot (\phi_1 \nabla \phi_2) \, d^3 \vec{r} = \iiint_V \phi_1 \nabla^2 \phi_2 \, d^3 \vec{r} \]

Now, let's go back to \( \vec{a} \), and take \( \phi_1 = \phi_2 = \phi \), with \( \phi^2 = -\vec{a} \cdot \vec{\theta} \).

We have (since \( \nabla^2 \phi = 0 \))

\[
\iiint_V (\vec{a} \cdot d^3 \vec{r} = \iiint_V \nabla \phi \cdot \dive \phi \, d^3 \vec{r} = \iint_S \phi \cdot \nabla \phi \cdot d\vec{S} = -\iint_S \phi \nabla^2 \phi \cdot d\vec{S} = 0
\]

Since \( \vec{a} \cdot d^3 \vec{r} = \vec{d} \cdot d^3 \vec{r} = \vec{d}^2 \geq 0 \) the only way for this to hold is that

\[
\vec{d} = 0 \Rightarrow \nabla \vec{v}_1 = \nabla \vec{v}_2 \Rightarrow \nabla \vec{v}_1 \text{ is unique}.
\]

Now let's show Helmholtz' theorem:

"A vector \( \vec{v} \) with divergence \( \theta \) and vorticity \( \omega \) vanishing at infinity can be written as the sum of two flows, one potential and the other rotational," i.e.,

\[
\begin{align*}
\nabla \cdot \vec{v} &= \theta (r) \\
\nabla \times \vec{v} &= \omega (r)
\end{align*} \Rightarrow \vec{v} = -\nabla \phi + \nabla \times \vec{A}
\]
\[
\begin{align*}
\phi(\mathbf{r}) & = \frac{1}{4\pi} \int \frac{\Theta(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d^3r' \\
\mathbf{A}(\mathbf{r}) & = \frac{1}{4\pi} \int \frac{\mathbf{\omega}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d^3r'
\end{align*}
\]

We can see immediately that we should have

\[
\begin{align*}
\nabla \cdot \mathbf{v} & = -\nabla^2 \phi = \Theta \quad \text{(since } \nabla \cdot (\nabla \times \mathbf{A}) = 0) \\
\nabla \times \mathbf{v} & = \nabla \times (\nabla \times \mathbf{A}) = \mathbf{\omega} \quad \text{(since } \nabla \times \nabla \phi = 0)
\end{align*}
\]

So, indeed \( \Theta \) determines \( \phi \), \( \mathbf{\omega} \) determines \( \mathbf{A} \).

Let's verify the scalar potential part first:

\[
-\nabla^2 \phi = -\nabla^2 \frac{1}{4\pi} \int \frac{\Theta(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d^3r'
\]

Note that the Laplacian acts on variables \( \mathbf{r} \), not \( \mathbf{r}' \). So we need to calculate

\[
\nabla^2 \left( \frac{1}{|\mathbf{r}|} \right) = ?
\]

Let's take \( \mathbf{F}' = 0 \) for the moment, and calculate \( \nabla^2 \left( \frac{1}{r} \right) \) - using spherical coordinates, we obtain:

\[
\nabla^2 \left( \frac{1}{r} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} \left[ r^2 \frac{\partial}{\partial r} \left( \frac{1}{r} \right) \right] = \frac{1}{r^2} \frac{\partial}{\partial r} \left[ r^2 \left( -\frac{1}{r^2} \right) \right]
\]

Now, if \( r \to 0 \) we have:

\[
\nabla^2 \left( \frac{1}{r} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} \left[ -1 \right] = 0
\]

What happens if \( r = 0 \)? To examine this, let's calculate,
\[ \int_V \nabla^2 \left( \frac{1}{r} \right) \, d^3r = \int_V \nabla \cdot \nabla \left( \frac{1}{r} \right) \, d^3r = \int_{\partial V} \nabla \left( \frac{1}{r} \right) \cdot \hat{r} \, ds \]

Volume about the origin \( r=0 \)

Now, \( \nabla \left( \frac{1}{r} \right) = -\frac{1}{r^2} \hat{r} \), and let's assume \( V \) is a sphere very very close to \( r=0 \) with radius \( R \to 0 \)

\[ \Rightarrow \int = \int_{\partial V} \frac{-\hat{r}}{R^2} \cdot \hat{r} \, ds = -\int \frac{\hat{r}}{R} \cdot \hat{r} \, ds = -\frac{1}{R^2} \int_{\partial V} ds = -\frac{1}{R^2} 4\pi R^2 \]

\[ \Rightarrow \int_{r=0} V^2 \left( \frac{1}{r} \right) \, d^3r = -4\pi \]

So, although \( V^2 \left( \frac{1}{r} \right) = 0 \) for all \( r \neq 0 \), its integral about \( r=0 \) is not zero. The only way this can happen, is if the "function" \( V^2 \left( \frac{1}{r} \right) \) is infinite @ \( r=0 \), but in such a way that its volume integral is finite. Such a thing is called the Dirac delta function \( \delta(r) \),

\[ \nabla^2 \left( \frac{1}{r} \right) = -4\pi \delta_0 \]

With properties:

\[ \delta_0 = 0 \quad \text{if} \quad r \neq 0 \quad \text{(} \delta_0 \text{ is infinite)} \]

\[ \int_V \delta_0 \, d^3r = 1 \quad \text{if} \quad V \text{ includes } r = 0 \]

\[ \int_V \delta_0 f(r) \, d^3r = f(0) \quad \text{evaluate function @ } r=0 \]

or

\[ \int \delta_0 f(r) \, d^3r = f(0) \quad \text{evaluate } f \text{ @ } r=0, \text{ zero argument for } \delta_0 \]

This we will study the delta function in detail next class. Let us proceed now with our calculation. We have then,
\[ \nabla^2 \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = -4\pi \delta(\mathbf{r} - \mathbf{r}') \]

\[
\Rightarrow -\nabla^2 \phi(\mathbf{r}) = -\nabla^2 \frac{1}{4\pi} \int \frac{\Theta(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d^3r' = -\frac{1}{4\pi} \int \nabla^2 \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \Theta(\mathbf{r}') \, d^3r' \\
= -\frac{1}{4\pi} \int -4\pi \delta(\mathbf{r} - \mathbf{r}') \Theta(\mathbf{r}') \, d^3r' = \int \delta(\mathbf{r} - \mathbf{r}') \Theta(\mathbf{r}') \, d^3r' \leq \Theta(\mathbf{r})
\]

\[
\Rightarrow \text{indeed we have } \nabla \cdot \nabla \phi = -\nabla^2 \phi = \Theta(\mathbf{r}).
\]

For \( A \) works similarly:

\[
\nabla \times \nabla \phi = \nabla \cdot (\nabla \times \mathbf{A}) = \nabla \cdot \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \equiv \nabla \times \omega
\]

Now it can be shown from definition of \( \mathbf{A} \) in terms of \( \omega \), that \( \nabla \cdot (\nabla \cdot \mathbf{A}) = 0 \), by integrating by parts, using Gauss's theorem, the fact that \( \nabla \cdot \omega = 0 \) and that \( \omega \) vanishes at infinity \( \Rightarrow \nabla \cdot (\nabla \cdot \mathbf{A}) = 0 \) (Try it!)

So,

\[-\nabla^2 \mathbf{A} = \omega \]

is what we need to show.

But,

\[
-\nabla^2 \mathbf{A} = -\frac{1}{4\pi} \nabla^2 \int \frac{\omega(\mathbf{r}') \, d^3r'}{|\mathbf{r} - \mathbf{r}'|} = \int \omega(\mathbf{r}') \delta(\mathbf{r} - \mathbf{r}') \, d^3r' = \omega(\mathbf{r})
\]

Therefore I can think of any complicated vector field of being locally a rotating flow + a potential flow (source + sinks).