The Dirac Delta Function

Relevant Material: Sniadecki Ch 13
Baas Ch 15 Sect 7

[I will follow Sniadecki here]

In vector and matrix algebra one has the identity matrix

\[ \mathbf{1} : \mathbf{v} = \mathbf{v} \]

\[ (\mathbf{1})_{ij} = \delta_{ij} \quad \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \]

In components,

\[ \sum_j (\mathbf{1})_{ij} v_j = v_i \]

The idea is to generalize this concept to continuous functions, rather than discrete indices \( ij \) - we look for a "function" \( \delta \) such that

\[ \int f(\mathbf{x}) \delta(\mathbf{x-x_0}) \, d\mathbf{x} = f(x_0) \]

A naive approach would be to take \( \delta(x-x_0) = \frac{1}{\pi} \frac{1}{x-x_0} \), but this doesn't work, because the integrand will be zero except at \( x=x_0 \) where it is \( f(x_0) \), but such an integral is zero, because a single point with finite value gives a vanishing contribution to an integral.

The only way to make such a thing pick up the value of \( f \) at \( x=x_0 \), is to make \( \delta(\mathbf{x-x_0}) = \infty \), so we can get a finite contribution to the integral. From this it is clear that this \( \delta \) will not be a function in the usual sense; mathematicians call the "function" we are about to define "distributions". Note that although they are infinite @ \( x=x_0 \), the result of integrals over it will be finite always. So things are well-defined under \( \int \) s.
In order to proceed we will define such a "function" as a limiting procedure. Consider the boxcar function

$$B_a(x) = \begin{cases} \frac{1}{2a} & 1 \leq x \leq a \\ 0 & \text{otherwise} \end{cases}$$

which depends on parameter $a$. As $a \to 0$ this will go to the desired function, note that it gets higher @ $x=0$, and it is zero almost everywhere (except @ $x=0$). Note that,

$$\int_{-\infty}^{\infty} B_a(x) \, dx = \int_{-a}^{a} \frac{1}{2a} \, dx = \frac{1}{2a} \int_{-a}^{a} x \, dx = \frac{1}{2a} \left[ \frac{1}{2} x^2 \right]_{-a}^{a} = \frac{1}{2a} (a - (-a)) = 1$$

So, as $a \to 0$ the area under $B_a(x)$ is always unity. Furthermore,

$$\int_{-\infty}^{\infty} B_a(x-x_0) f(x) \, dx = \int_{-\infty}^{\infty} B_a(y) f(y+x_0) \, dy = \frac{1}{2a} \int_{-a}^{a} f(y+x_0) \, dy = \frac{1}{2a} \int_{x_0-a}^{x_0+a} f(x) \, dx = \text{average of } f(x) \text{ about } x=x_0$$

Note that indeed the limit integral is only dependent on $f(x)$ near $x=x_0$ if $a$ is small, and it corresponds to the average of $f(x)$ near $x=x_0$, by summing over the values in interval of width $2a$ (from $x_0-a$ to $x_0+a$) and dividing by $2a$.

As $a \to 0$ the average of the function can only be $f(x_0)$, i.e.

$$\lim_{a \to 0} \int_{-\infty}^{\infty} B_a(x-x_0) f(x) \, dx = f(x_0) = \int_{-\infty}^{\infty} \delta_0(x-x_0) f(x) \, dx$$

such a limit defines what we call as the Dirac delta function $\delta_0(x-x_0)$ centered at point $x=x_0$, i.e. $\lim_{a \to 0} B_a(x-x_0) \to \delta_0(x-x_0)$
We can see that indeed such definition leads to $f(\gamma_0)$.

We need to show that

$$
\alpha = \lim_{a \to 0} \frac{1}{2a} \int_{\gamma_0-a}^{\gamma_0+a} f(x) \, dx = f(\gamma_0)
$$

Let's expand in Taylor series about $x = \gamma_0$, since the integral is dominated by contributions at $x = \gamma_0$,

$$
f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{d^n f}{dx^n} \right)(\gamma_0) (x-\gamma_0)^n
$$

$$
= f(\gamma_0) + f'(\gamma_0) (x-\gamma_0) + f''(\gamma_0) \frac{(x-\gamma_0)^2}{2} + \ldots
$$

$$
\Rightarrow \alpha = \lim_{a \to 0} \frac{1}{2a} \int_{\gamma_0-a}^{\gamma_0+a} \left[ f(\gamma_0) + f'(\gamma_0) (x-\gamma_0) + \ldots \right] \, dx
$$

Now, the terms in Taylor series have dependence on $x$ only through $(x-\gamma_0)^n$; so we need to compute the integrals

$$
\int_{\gamma_0-a}^{\gamma_0+a} (x-\gamma_0)^n \, dx = \left. \frac{(x-\gamma_0)^{n+1}}{n+1} \right|_{\gamma_0-a}^{a} = \frac{a^{n+1} - (-a)^{n+1}}{n+1}
$$

$$
= \begin{cases} 
0 & n \text{ odd} \\
\frac{2}{n+1} a^{n+1} & n \text{ even}
\end{cases}
$$

Then we have,

$$
\alpha = \lim_{a \to 0} \frac{1}{2a} \left[ 2a f(\gamma_0) + \frac{2a^3}{3} + \frac{1}{2} f''(\gamma_0) + \ldots \right]
$$

$$
= \lim_{a \to 0} \left[ f(\gamma_0) + \frac{1}{6} a^2 f''(\gamma_0) + \text{high powers of } a \right]
$$

$$
= f(\gamma_0) \checkmark
$$
The delta function can be defined as well as a limit of other functions (we shall see this later), not only $\theta_{a}(x)$ as $a \to 0$. In all cases it involves of course a function that becomes infinite at $x=0$ but whose area is constant (and equal to unity) as the parameter approaches its limit (e.g. $a \to 0$).

So, the main properties are,

\[
\int_{-\infty}^{\infty} \delta(x-x_0) \, dx = 1
\]

\[
\int_{-\infty}^{\infty} dx \, \delta(x-x_0) \, f(x) = f(x_0)
\]

In addition,

\[
\int_{-\infty}^{\infty} dx \, \delta_{\frac{c}{C}}[c(x-x_0)] \, f(x) = \frac{f(x_0)}{|c|}
\]

we can see this easily. Assume $c > 0$, then,

\[
\int_{-\infty}^{\infty} dx \, \delta_{\frac{c}{C}}[c(x-x_0)] \, f(x) = \int_{-\infty}^{\infty} dy \, \delta_{\frac{c}{C}}[y-cx_0] \, f\left(\frac{y}{c}\right)
\]

\[
= \frac{1}{c} \int_{-\infty}^{\infty} dy \, \delta_{\frac{c}{C}}[y-x_0] \, f\left(\frac{y}{c}\right) = \frac{1}{c} \int_{-\infty}^{\infty} dy \, \delta_{\frac{c}{C}}[y-cx_0] \, f\left(\frac{y}{c}\right)
\]

\[
y = cx_0
\]

For $c < 0$ we have similarly,

\[
\int_{-\infty}^{\infty} dx \, \delta_{\frac{c}{C}}[c(x-x_0)] \, f(x) = \int_{-\infty}^{\infty} dy \, \delta_{\frac{c}{C}}[y-cx_0] \, f\left(\frac{y}{c}\right) = -\frac{1}{c} \int_{-\infty}^{\infty} dy \, \delta_{\frac{c}{C}}[y-\frac{x_0}{c}] \, f\left(\frac{y}{c}\right)
\]

Note that since $c < 0$, when $x \to +\infty$, $y \to -\infty$ and vice versa, thus the change in limit of integration. Then,
\[
\int_0^\infty dx \, \delta_c [c(x-x_0)] \, f(x) = -\frac{1}{c} \, f(x_0)
\]

de so indeed
\[
\int_0^\infty dx \, \delta_c [c(x-x_0)] \, f(x) = \frac{f(x_0)}{1c}
\]

In other words,
\[
\delta_c [c(x-x_0)] = \frac{\delta_c (x-x_0)}{1c}
\]

Similarly, we can calculate the delta function of a function, e.g.
\[
\int \delta_c [g(x)] \, f(x) \, dx = ?
\]

Let \( g \) be zero for some \( x=x_0 \) (\( g(x_0)=0 \)) — obviously the \( \delta_c (x-x_0) \) will pick up the value of \( f(x) \) at \( x=x_0 \), that is
\[
\int \delta_c [g(x)] \, f(x) \, dx = A \, f(x_0) \quad \text{where} \quad g(x_0)=0
\]

for some proportionality constant \( A \), which we have to calculate.

Again, we can expand \( g(x) \) about \( x=x_0 \), since integrated will be dominated by contribution near \( x=x_0 \),
\[
g(x) = g(x_0) + g'(x_0) (x-x_0) + g''(x_0) \frac{(x-x_0)^2}{2} + \ldots
\]

\( \approx g'(x_0) (x-x_0) + \text{higher-order terms} \)

Then to first approximation,
\[
\int \delta_c [g(x)] \, f(x) \, dx = \int \delta_c [g'(x_0) (x-x_0)] \, f(x) \, dx
\]
But now this is exactly as the previous case for \(c = g'(x_0)\), since \(g'(x_0)\) is a constant independent of \(x\), then

\[
\frac{\int_{-\infty}^{\infty} \delta_{g(x)} f(x) \, dx}{g'(x_0)} = \frac{f(x_0)}{|g'(x_0)|}
\]

or:

\[
\delta_{g(x)} = \frac{\delta(x-x_0)}{|g'(x_0)|}
\]

where \(x_0\) is such that \(g(x_0) = 0\).

Now, the function \(g(x)\) may have lots of zeros, then each one contributes as well. If \(g(x_i) = 0\) \(i = 1, ..., N\)

\[
\Rightarrow \delta_{g(x)} = \sum_{i=1}^{N} \frac{\delta(x-x_i)}{|g'(x_i)|}
\]

\(\delta_0\)'s for we discussed things in 1D. In more than one dimension, things behave as you expect, e.g., in 3D we have:

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x-x_0) \delta(y-y_0) \delta(z-z_0) f(x, y, z) \, dx \, dy \, dz
\]

that is, a delta function in each dimension. So then

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x-x_0) \delta(y-y_0) \delta(z-z_0) f(x, y, z) \, dx \, dy \, dz
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(y-y_0) \delta(z-z_0) f(x_0, y, z) \, dy \, dz = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_0, y_0, z) \delta(z-z_0) \, dy \, dz = f(x_0, y_0, z_0)
\]

\[
= f(F_0)
\]
And we also have of course: \[ \int \delta_D(\mathbf{r} - \mathbf{r}_0) \, d^3r = 1 \]

Since this follows from previous work by letting \( f(\mathbf{r}) = 1 \).

In spherical coordinates, it is a bit more tricky. For example on the surface of the sphere, we want that

\[ \int dS \cdot \delta_D(\mathbf{r} - \mathbf{r}_0) \cdot f(\mathbf{r}) = f(\mathbf{r}_0) \]

\[ dS = \sin \theta \, d\theta \, d\varphi \]

where \( \mathbf{r} \) is the unit vector that points in direction \((\theta, \varphi)\):

\[ \mathbf{r} = \left( \begin{array}{c} \sin \theta \cos \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{array} \right) \]

and similarly for \( \mathbf{r}_0 \).

\[ \int dS \cdot \delta_D(\mathbf{r} - \mathbf{r}_0) \cdot f(\mathbf{r}) = \int_0^\pi \sin \theta \, d\theta \int_0^{2\pi} d\varphi \ \delta_D(\theta - \theta_0) \delta_D(\varphi - \varphi_0) \ f(\theta, \varphi) \]

where we wrote \( \delta_D(\mathbf{r} - \mathbf{r}_0) = \delta_D(\theta - \theta_0) \delta_D(\varphi - \varphi_0) \), i.e. a delta function in each dimension times a constant we have to figure out.

We see that to recover \( f(\theta_0, \varphi_0) \), we need \( A \sin \theta = 1 \), for then

\[ \int dS \cdot \delta_D(\mathbf{r} - \mathbf{r}_0) \cdot f(\mathbf{r}) = \int_0^\pi \delta_D(\theta - \theta_0) \int_0^{2\pi} \delta_D(\varphi - \varphi_0) \ f(\theta, \varphi) = f(\theta_0, \varphi_0) = f(\mathbf{r}_0) \]

Then we have in the surface of the sphere,

\[ \delta_D(\mathbf{r} - \mathbf{r}_0) = \frac{\delta_D(\theta - \theta_0) \ \delta_D(\varphi - \varphi_0)}{\sin \theta} \quad \text{(Spherical 2D)} \]

We can similarly work out the 3D case in spherical coordinates:
We want
\[ \int \delta_0 (\mathbf{r} - \mathbf{r}_0) f(\mathbf{r}) \, d^3r = f(\mathbf{r}_0) \]

But in spherical coordinates, \( d^3r = r^2 \sin \theta \, dr \, d\theta \, d\phi \)

Let's write again \( \delta_0 (\mathbf{r} - \mathbf{r}_0) = A \, \frac{\delta(r - r_0)}{r} \, \frac{\delta(\theta - \theta_0)}{\sin \theta} \, \frac{\delta(\phi - \phi_0)}{2\pi} \)

Then:
\[ \int A r^2 \sin \theta \, \delta_0 (r - r_0) \, \delta_0 (\theta - \theta_0) \, \delta_0 (\phi - \phi_0) \, f(r_1, \theta_1, \phi_1) \, dr \, d\theta \, d\phi = f(r_0, \theta_0, \phi_0) \]

\( \Rightarrow \) \( Ar^2 \sin \theta = 1 \)

So that:
\[ \delta_0 (\mathbf{r} - \mathbf{r}_0) = \frac{\delta(r - r_0)}{r^2 \sin \theta} \, \frac{\delta(\theta - \theta_0)}{\sin \theta} \, \frac{\delta(\phi - \phi_0)}{2\pi} \]